Sporadic Groups and Conformal Field Theories

Jeff Harvey
University of Chicago

Exceptional Dimensions
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My first introduction to Bernard's work occurred when I learned about the Julia-Zee dyon while working on magnetic monopoles and fermions, inspired by the work of Callan and Rubakov. It was only later★ that I learned to appreciate his work on infinite-dimensional symmetries, supergravities and exceptional things.

Today I want to discuss two related topics that I hope will appeal to his love of exceptional things:

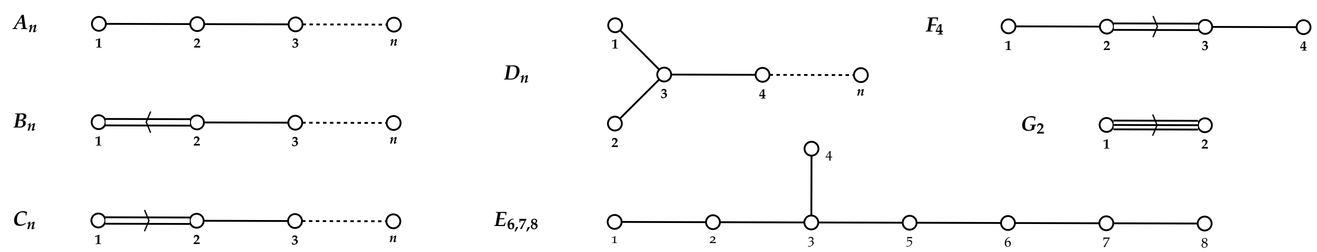
1. The existence of sporadic group symmetry in special 2d CFTs (with J-B Bae, K. Lee, S. Lee and B. Rayhaun)

2. A connection between superconformal symmetry, quantum error correction, and sporadic group symmetry (with G. Moore).

★ This is probably wrong. Cremmer&Julia->Gell-Mann->Ramond

The Periodic Table Of Finite Simple Groups

Dynkin Diagrams of Simple Lie Algebras



$0, C_1, \mathbb{Z}_1$ 1 1																	C_2 2
$A_1(4), A_1(5)$ A_5 60	$A_2(2)$ $A_1(7)$ 168											$^2A_3(4)$ $B_2(3)$ 25 920	$C_3(3)$ 4 585 351 680	$D_4(2)$ 174 182 400	$^2D_4(2^2)$ 197 406 720	$G_2(2)'$ $^2A_2(9)$ 6 048	C_3 3
$A_1(9), B_2(2)'$ A_6 360	$^2G_2(3)'$ $A_1(8)$ 504											$B_2(4)$ 979 200	$C_3(5)$ 228 501 000 000 000	$D_4(3)$ 4 952 179 814 400	$^2D_4(3^2)$ 10 151 968 619 520	$^2A_2(16)$ 62 400	C_5 5
A_7 2 520	$A_1(11)$ 660	$E_6(2)$ 214 841 575 522 005 575 270 400	$E_7(2)$ 7 997 476 042 075 799 759 100 487 262 680 802 918 400	$E_8(2)$ 337 804 753 143 634 806 261 388 190 614 085 595 079 991 692 242 467 651 576 160 959 909 068 800 000	$F_4(2)$ 3 311 126 603 366 400	$G_2(3)$ 4 245 696	$^3D_4(2^3)$ 211 341 312	$^2E_6(2^2)$ 76 532 479 683 774 853 939 200	$^2B_2(2^3)$ 29 120	Tits* $^2F_4(2)'$ 17 971 200	$^2G_2(3^3)$ 10 073 444 472	$B_3(2)$ 1 451 520	$C_4(3)$ 65 784 756 654 489 600	$D_5(2)$ 23 499 295 948 800	$^2D_5(2^2)$ 25 015 379 558 400	$^2A_2(25)$ 126 000	C_7 7
$A_3(2)$ A_8 20 160	$A_1(13)$ 1 092	$E_6(3)$ 7 257 703 347 541 463 210 028 258 395 214 643 200	$E_7(3)$ 1 271 375 236 818 136 742 240 479 751 139 021 644 554 379 203 770 766 254 617 395 200	$E_8(3)$ 18 830 052 912 953 932 311 099 032 439 972 660 332 140 886 784 940 152 038 522 449 391 826 616 580 150 109 878 711 243 949 982 163 694 448 626 420 940 800 000	$F_4(3)$ 5 734 420 792 816 671 844 761 600	$G_2(4)$ 251 596 800	$^3D_4(3^3)$ 20 560 831 566 912	$^2E_6(3^2)$ 14 636 855 916 969 695 633 965 120 680 532 377 600	$^2B_2(2^5)$ 32 537 600	$^2F_4(2^3)$ 264 905 352 699 586 176 614 400	$^2G_2(3^5)$ 49 825 657 439 340 552	$B_2(5)$ 4 680 000	$C_3(7)$ 273 457 218 604 953 600	$D_4(5)$ 8 911 539 000 000 000 000	$^2D_4(4^2)$ 67 536 471 195 648 000	$^2A_3(9)$ 3 265 920	C_{11} 11
A_9 181 440	$A_1(17)$ 2 448	$E_6(4)$ 85 528 710 781 342 640 103 833 619 055 142 765 466 746 880 000	$E_7(4)$ 111 131 458 114 940 385 379 597 233 477 884 941 280 664 199 527 155 056 307 251 745 263 504 588 800 000 000	$E_8(4)$ 191 797 292 142 671 717 754 639 757 897 512 906 421 357 507 604 216 557 533 558 287 598 236 977 154 127 870 984 484 770 345 340 348 298 409 697 395 609 822 849 492 217 656 441 474 908 160 000 000 000	$F_4(4)$ 19 009 825 523 840 945 451 297 669 120 000	$G_2(5)$ 5 859 000 000	$^3D_4(4^3)$ 67 802 350 642 790 400	$^2E_6(4^2)$ 85 696 576 147 617 709 485 896 772 387 584 983 695 360 000 000	$^2B_2(2^7)$ 34 093 383 680	$^2F_4(2^5)$ 1 318 633 155 799 591 447 702 161 609 782 722 560 000	$^2G_2(3^7)$ 239 189 910 264 352 349 332 632	$B_2(7)$ 138 297 600	$C_3(9)$ 54 025 731 402 499 584 000	$D_5(3)$ 1 289 512 799 941 305 139 200	$^2D_4(5^2)$ 17 880 203 250 000 000 000	$^2A_2(64)$ 5 515 776	C_{13} 13
A_n $\frac{n!}{2}$	$PSL_{n+1}(q), L_{n+1}(q)$ $A_n(q)$ $\frac{q^{n(n+1)/2}}{(n+1,q-1)}\prod_{i=1}^{n}(q^{i+1}-1)$	$E_6(q)$ $\frac{q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)}{(3,q-1)}$	$E_7(q)$ $\frac{q^{63}}{(2,q-1)}\prod_{\substack{i=1\\ i\neq 2,8}}^{9}(q^{2i}-1)$	$E_8(q)$ $q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$	$F_4(q)$ $q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$	$G_2(q)$ $q^6(q^6-1)(q^2-1)$	$^3D_4(q^3)$ $q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	$^2E_6(q^2)$ $\frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{(3,q+1)}$	$^2B_2(2^{2n+1})$ $q^2(q^2+1)(q-1)$	$^2F_4(2^{2n+1})$ $q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	$^2G_2(3^{2n+1})$ $q^3(q^3+1)(q-1)$	$O_{2n+1}(q), \Omega_{2n+1}(q)$ $B_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$PSp_{2n}(q)$ $C_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$O_{2n}^{+}(q)$ $D_n(q)$ $\frac{q^{n(n-1)}(q^n-1)}{(4,q^n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$O_{2n}^{-}(q)$ $^2D_n(q^2)$ $\frac{q^{n(n-1)}(q^n+1)}{(4,q^n+1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$PSU_{n+1}(q)$ $^2A_n(q^2)$ $\frac{q^{n(n+1)/2}}{(n+1,q+1)}\prod_{i=2}^{n+1}(q^i-(-1)^i)$	$\mathbb{Z}_p$ C_p p

- Alternating Groups
- Classical Chevalley Groups
- Chevalley Groups
- Classical Steinberg Groups
- Steinberg Groups
- Suzuki Groups
- Ree Groups and Tits Group*
- Sporadic Groups
- Cyclic Groups

Alternates†
Symbol
Order‡

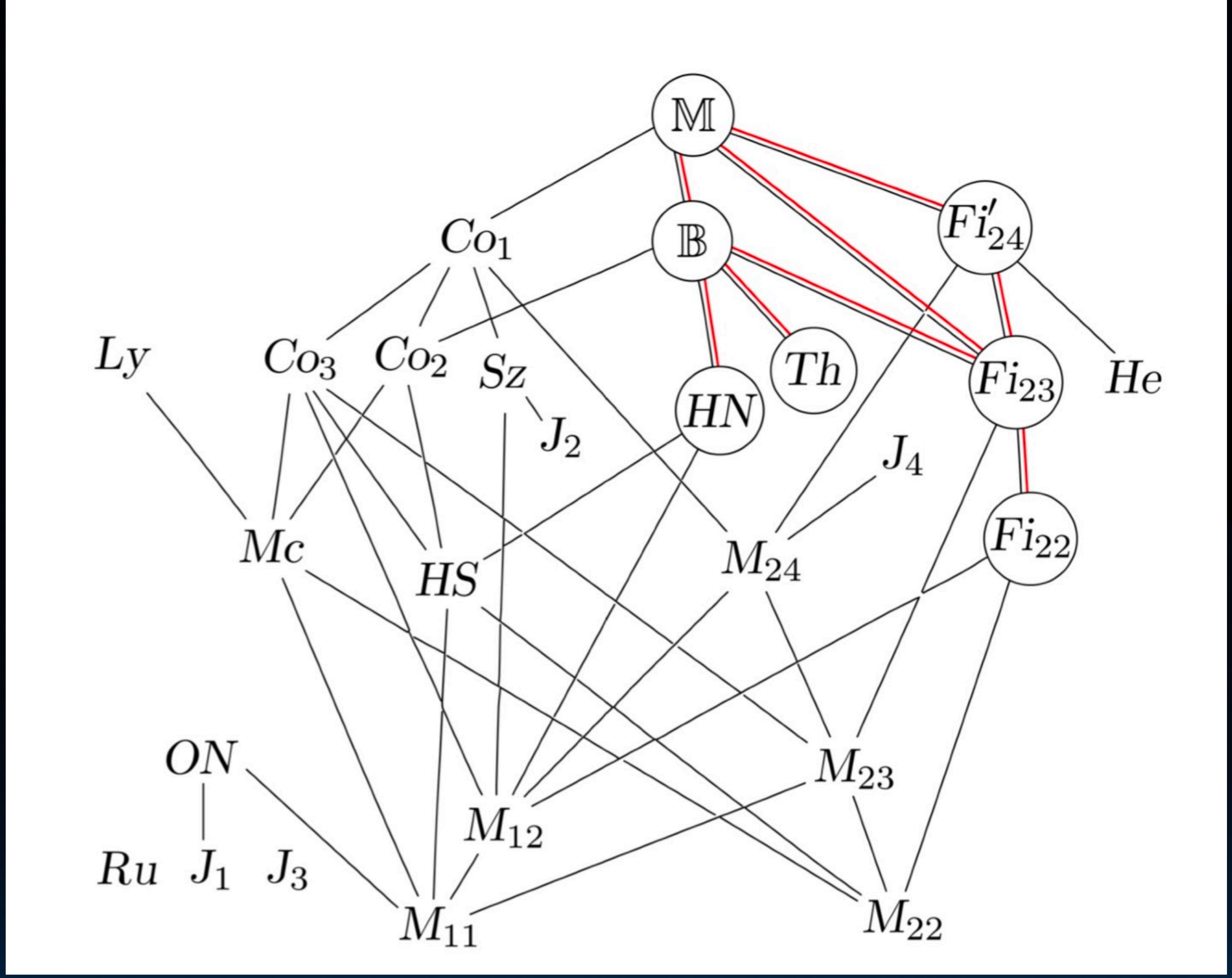
M_{11} 7 920	M_{12} 95 040	M_{22} 443 520	M_{23} 10 200 960	M_{24} 244 823 040	$J(1), J(11)$ J_1 175 560	HJ J_2 604 800	HJM J_3 50 232 960	J_4 86 775 571 046 077 562 880	HS 44 352 000	McL 898 128 000	F_7, HHM, HTH He 4 030 387 200	Ru 145 926 144 000
Sz Suz 448 345 497 600	$O'NS, O\text{–}S$ $O'N$ 460 815 505 920	$\cdot 3$ Co_3 495 766 656 000	$\cdot 2$ Co_2 42 305 421 312 000	$\cdot 1$ Co_1 4 157 776 806 543 360 000	F_5, D HN 273 030 912 000 000	LyS Ly 51 765 179 004 000 000	F_3, E Th 90 745 943 887 872 000	$M(22)$ Fi_{22} 64 561 751 654 400	$M(23)$ Fi_{23} 4 089 470 473 293 004 800	$F_{3+}, M(24)'$ Fi_{24}' 1 255 205 709 190 661 721 292 800	F_2 B 4 154 781 481 226 426 191 177 580 544 000 000	F_1, M_1 M 808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000

*The Tits group $^2F_4(2)'$ is not a group of Lie type, but is the (index 2) commutator subgroup of $^2F_4(2)$. It is usually given honorary Lie type status.

†For sporadic groups and families, alternate names in the upper left are other names by which they may be known. For specific non-sporadic groups these are used to indicate isomorphims. All such isomorphisms appear on the table except the family $B_n(2^m) \cong C_n(2^m)$.

The groups starting on the second row are the classical groups. The sporadic suzuki group is unrelated to the families of Suzuki groups.

‡Finite simple groups are determined by their order with the following exceptions:
$B_n(q)$ and $C_n(q)$ for q odd, $n > 2$;
$A_8 \cong A_3(2)$ and $A_2(4)$ of order 20160.



20 of the 26 sporadic groups are embedded as quotients of subgroups of the Monster group. 6 (ON, Ru, Ly, J1, J3, J4) are not.

Groups can of course be studied abstractly, but in physics (and often in math) we would like to know if they acts as the symmetry or automorphism group of some object, preserving some structure.

S_4

A_5

A_5

For the sporadic groups there is evidence that the right structures are those of 2d Conformal Field Theories (CFTs) or VOAs with the group preserving the algebraic structure of the operator product expansion.

We know that there is a c=24 CFT with Monster symmetry that explains the original moonshine connection between the Monster and modular forms. It is a Z2 orbifold of the Leech lattice CFT/VOA.

$$\mathrm{Tr} q^{L_0 - c/24} = J(\tau) = q^{-1} + (196883 + 1)q + \cdots$$

Are the sporadic groups appearing in M also the symmetry groups of CFTs embedded in the Monster CFT?

A key observation needed to answer this question was made by Zamolodchikov and generalized by Dixon and JH, Dong-Mason-Zhu, Dong-Li-Mason-Norton.

A CFT always has a stress tensor. If it also has primary dimension (2,0) fields it can have other stress tensors (conformal vectors) with smaller central charge. For example, with a single (2,0) primary we have the OPEs

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

$$T(z)\varphi(w) \sim \frac{2\varphi(w)}{(z-w)^2} + \frac{\partial\varphi(w)}{z-w}$$

$$\varphi(z)\varphi(w) \sim \frac{1}{(z-w)^4} + \frac{4T(w)/c + b\varphi(w)}{(z-w)^2} + \frac{\partial\big(4T(w)/c + b\varphi(w)\big)/2}{z-w}.$$

Write $t(z) = \alpha T(z) + \beta\varphi(z)$ **and demand that**

$$t(z)t(w) \sim \frac{c_t/2}{(z-w)^4} + \frac{2t(w)}{(z-w)^2} + \cdots$$ **solve for** α, β

One finds two solutions and this leads to a decomposition into sub CFTs $T(z) = t_+(z) + t_-(z)$ $\quad c = c_+ + c_-$

This is similar to the GKO coset construction but does not involve any currents (dimension (1,0) operators).

Example 1: c=1 CFT on a circle of radius R=1

$$\varphi(z) = \frac{1}{\sqrt{2}}\left(e^{2iX(z)} + e^{-2iX(z)}\right)$$

$$t_{\pm} = \frac{1}{2}\left(-\frac{1}{2}\partial X \partial X \pm \frac{1}{\sqrt{2}}\varphi\right)$$

c=1/2 stress tensors (conformal vectors)

Example 2: Lattice CFT/VOA with lattice $\sqrt{2}R$, R a root lattice of type A_n, D_n, E_6, E_7, E_8. Because of rescaling the roots now correspond to dimension 2 operators.

There is a linear combination of T and φ which is a conformal vector with

$$\varphi = \frac{1}{2h+2} \sum_{\alpha \in \Phi^+(R)} \left(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}\right)$$

R	c	
A_n	$\frac{2n}{n+3}$	$\mathbb{Z}_{n+1}\ pf$
D_n	1	
E_6	$6/7$	*minimal models*
E_7	$7/10$	
E_8	$1/2$	

Note that φ is invariant under X -> -X. And that the Leech lattice (with minimal length squared 4) contains many sub lattices of the form $\sqrt{2}R$. Therefore we can find many subCFTs of the Monster CFT and "deconstruct" it into CFTs with smaller central charge.

The symmetries of these subCFTs have an interesting relation to the Monster and its subgroups. The simplest example is due to G. Höhn.

$$24 = \frac{1}{2} + 23\frac{1}{2}$$

Ising CFT with Z2 symmetry

$\mathcal{W}$

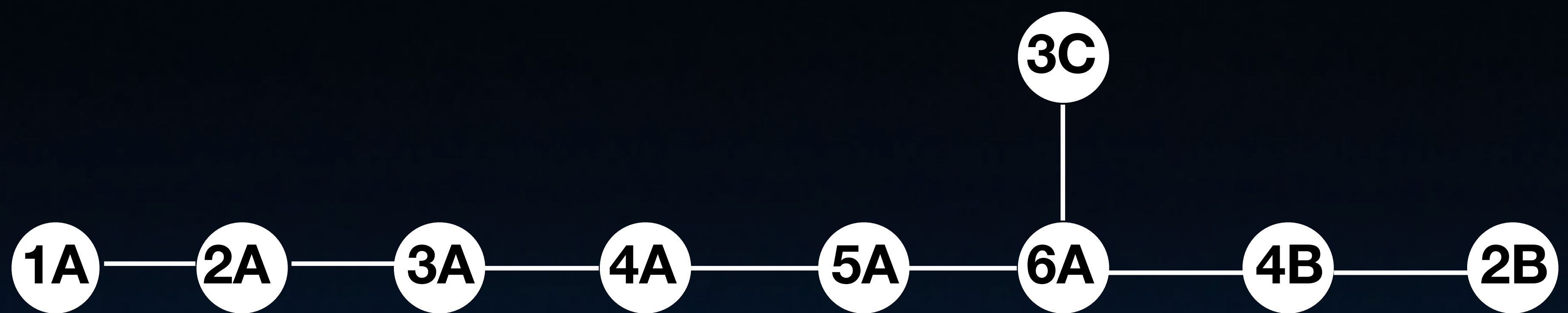
CFT with the Baby Monster =Centralizer(Z2) as its symmetry group

$\tilde{\mathcal{W}}$

More complicated examples have been studied in the math literature by Lam, Yamada, Yamauchi, Dong, Kitazume, Miyamoto, …

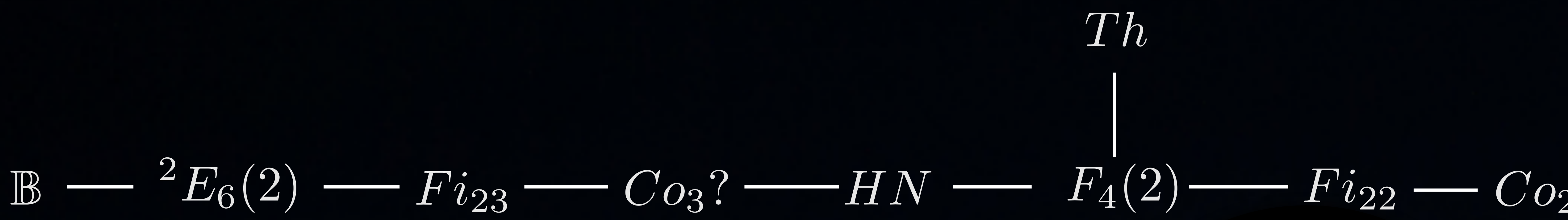
Elements in the 2A class of M $\longleftrightarrow \mathbb{Z}_2$ Symmetry of Ising Model

Products of 2A elements are elements of one of the 9A classes 1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A and 1,2,2,3,3,4,4,5,6 and are the Coxeter labels of the extended E8 Dynkin diagram (McKay)



$$\alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_3 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 = 0$$

It is then natural to ask whether the pairs of Ising models generate a sub-CFT $(\mathcal{W})$ of the Monster CFT, what it is, what is left when this sub-CFT is removed $(\tilde{\mathcal{W}})$, and what its symmetry group is. This produces a set of CFTs with sporadic symmetry groups.



In this way we can associated the nodes of the extended E8 Dynkin diagram to 7 sporadic groups and two groups of Lie type.

$\mathcal{W}$	$\widetilde{\mathcal{W}}$	$\mathrm{Aut}(\widetilde{\mathcal{W}})$	$\mathrm{Inn}(\widetilde{\mathcal{W}})$	$c_{\tilde{t}}$	subalgebra of
trivial	$V^\natural$	$\mathbb{M}$	$\mathbb{M}$	24	$V^\natural$
$\mathcal{W}_{1\mathrm{A}} \cong L(\frac{1}{2},0)$	$V\mathbb{B}^\natural$	$\mathbb{B}$	$\mathbb{B}$	$23\frac{1}{2}$	$V^\natural$
$P(3)$	$VF_{24}^\natural$	Fi_{24}	Fi'_{24}	$23\frac{1}{5}$	$V^\natural$
$\mathcal{W}_{2\mathrm{A}}$	$\widetilde{\mathcal{W}}_{2\mathrm{A}}$	$^2E_6(2).2$		$22\frac{4}{5}$	$V\mathbb{B}^\natural$, $V^\natural$
$\mathcal{W}_{3\mathrm{A}}$	$VF_{23}^\natural$	Fi_{23}	Fi_{23}	$22\frac{12}{35}$	$VF_{24}^\natural$, $V\mathbb{B}^\natural$, $V^\natural$
$\mathcal{W}_{4\mathrm{A}}$	$\widetilde{\mathcal{W}}_{4\mathrm{A}}$			22	
$\mathcal{W}_{5\mathrm{A}}$	$VHN^\natural$	$HN.2$	HN	$21\frac{5}{7}$	$V\mathbb{B}^\natural$, $V^\natural$
$\mathcal{W}_{6\mathrm{A}}$	$VF_{22}^\natural$	$Fi_{22}.2$	Fi_{22}	$21\frac{9}{20}$	$VF_{23}^\natural$, $VF_{24}^\natural$, $V\mathbb{B}^\natural$, $V^\natural$
$\mathcal{W}_{4\mathrm{B}}$	$\widetilde{\mathcal{W}}_{4\mathrm{B}}$	$F_4(2).2$		$22\frac{1}{10}$	$\widetilde{\mathcal{W}}_{2\mathrm{A}}$, $V\mathbb{B}^\natural$, $V^\natural$
$\mathcal{W}_{2\mathrm{B}} \cong L(\frac{1}{2},0)^{\otimes 2}$	$\widetilde{\mathcal{W}}_{2\mathrm{B}}$	$2^{22}.Co_2$	$2^{22}.Co_2$	22	$V\mathbb{B}^\natural$, $V^\natural$
$\mathcal{W}_{3\mathrm{C}}$	$VT^\natural$	Th	Th	$22\frac{6}{11}$	$V\mathbb{B}^\natural$, $V^\natural$

The most intricate and novel part of our analysis is the determination of the characters of these CFTs with sporadic group symmetry. We use a number of techniques (Hecke operators for vector-valued modular forms, Modular Linear Differential Equations, and products of the above with minimal model characters to produce characters of the "small" and "large" sub CFTs with characters $\chi_i(q)$, $X_i(q)$

$$\sum_i \chi_i(q) X_i(q) = J(q) = q^{-1} + 196884q + \cdots$$

I will spare you most of the gory details and just present some results for one example, the 3C example with Thompson group symmetry.

The product of the two Z_2 elements in the two Ising theories is order 3 and is in the 3C class of the Monster. They generate a c=1/2+11/12 CFT which can be viewed as the tensor product of the Ising model (m=3 minimal model) and the m=8 minimal model. The characters are products of these minimal model characters and have an alternate representation in terms of Z_9 parafermion characters. The $\tilde{\mathcal{W}}$ CFT has Thompson sporadic group symmetry with characters:

$$
\begin{aligned}
\chi_{VT^\natural(0)}(\tau) &= q^{-\frac{31}{33}}(1 + 30876q^2 + 2634256q^3 + 90061882q^4 + 1855967520q^5 \\
&\quad + 27409643240q^6 + 317985320008q^7 + 3064708854915q^8 + \cdots) \\
\chi_{VT^\natural(1)}(\tau) &= q^{\frac{29}{33}}(30628 + 3438240q + 132944368q^2 + 2954702008q^3 + 45976123126q^4 \\
&\quad + 554583175040q^5 + 5510740058664q^6 + 46939446922208q^7 + \cdots) \\
\chi_{VT^\natural(2)}(\tau) &= q^{\frac{17}{33}}(4123 + 961248q + 49925748q^2 + 1315392496q^3 + 22953663126q^4 \\
&\quad + 301143085728q^5 + 3193490344856q^6 + 28662439021248q^7 + \cdots) \\
\chi_{VT^\natural(3)}(\tau) &= q^{\frac{32}{33}}(61256 + 5955131q + 216162752q^2 + 4622827508q^3 + 70051197488q^4 \\
&\quad + 828481014062q^5 + 8106388952544q^6 + 68191291976248q^7 + \cdots) \\
\chi_{VT^\natural(4)}(\tau) &= q^{\frac{8}{33}}(248 + 147498q + 10107488q^2 + 308975512q^3 + 5936748000q^4 \\
&\quad + 83455971224q^5 + 932866634976q^6 + 8730997273664q^7 + \cdots)
\end{aligned}
\tag{3.117}
$$

Are dimensions of Th irreps

This gives a fairly uniform construction of CFTs with symmetry groups that are sporadic groups (or minor extensions of sporadic group) for 7 of the sporadic group in the Monster. What about the others?

We have partial results using other products of minimal models and parafermions for several other groups but they seem to require further deconstruction before we have the right CFT "on the nose." These include Held, Hall-Janko, Suzuki and a couple of the Mathieu groups.

It is notable that the characters we find for Th and for Mathieu do not seem to be related to the weight 1/2 (mock) modular forms that exhibit moonshine for these groups in the work of Rayhaun & JH and Eguchi, Ooguri and Tachikawa.

It now seems plausible that all sporadic groups appearing as sub-quotients of the Monster can be viewed as the symmetry groups of certain special CFTs although quite a bit more work is required to see if this is really the case.

It is not clear if the remaining 6 sporadic groups can be fit into this framework, but some of them have recently been associated with modular forms/moonshine, so even they may eventually be understood using something like 2d CFT.

I'd like to change to change topics somewhat and briefly discussion a novel aspect of some special CFTs with sporadic symmetry groups.

Some of these CFTs are actually superconformal, and the superconformal generator and the symmetries preserving it are connected to quantum error correcting codes.

I will discuss one example, but we (G. Moore and I) have analyzed two others and hope the considerations may be more general.

Example: A c=6 SCFT describing a K3 sigma model with symmetry group $\mathbb{Z}_2^8 : M_{20}$ (Gaberdiel, Taormina, Volpato, Wendland) $T_{D_4}/\mathbb{Z}_2$ $SO(4)^3 = SU(2)^6$

Recall the N=1 superconformal algebra expressed in terms of the OPE:

$$G(z)G(w) \sim \frac{\frac{\hat{c}}{4}}{(z-w)^3} + \frac{\frac{1}{2}T(w)}{z-w} + \cdots$$

$$T(z)G(w) \sim \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} + \cdots$$

$$T(z)T(w) \sim \frac{\frac{3}{4}\hat{c}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots$$

G(z) is a dimension 3/2 operator. It can be constructed as a very special sum of tensor products of operators with smaller conformal weight:

$$\frac{3}{2} = 6 \times \frac{1}{4}$$

The construction is related to both classical error correcting codes and quantum error correcting codes.

A classical error correcting code in a linear subspace of the tensor product of n copies of the finite field with $q = p^n$ elements, $\mathcal{C} \subset \mathbb{F}_q^n$.

Physicists mainly encounter $\mathbb{F}_p$ with p prime, integers mod p. We need $\mathbb{F}_4$

$\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$ 0 is additive identity, 1 is multiplicative identity and the addition and multiplication rules are

$$
\begin{aligned}
\underline{1} + \underline{\omega} &= \underline{\bar{\omega}} \\
\underline{1} + \underline{\bar{\omega}} &= \underline{\omega} \\
\underline{\omega} + \underline{\bar{\omega}} &= \underline{1} \\
x + x &= \underline{0} \,.
\end{aligned}
$$

$$
\begin{aligned}
\underline{\omega\omega} &= \underline{\bar{\omega}} \\
\underline{\omega\bar{\omega}} &= \underline{1} \\
\underline{\bar{\omega}\bar{\omega}} &= \underline{\omega}
\end{aligned}
$$

Consider the GTVW K3 sigma model. It has a description in terms of a (subspace) of the tensor product of 6 level one affine SU(2) theories, $\widehat{SU(2)}_1$. This theory has a primary operator of dimension h=1/4 which occurs with multiplicity 2 and creates states in the 2-dimensional irrep of SU(2).

$$e^{\frac{i}{\sqrt{2}}\epsilon X(z)}, \quad \epsilon = \pm 1$$

The product of six such operators has dimension 6/4=3/2 and can be labelled by

$$(\epsilon_1, \epsilon_2, \cdots \epsilon_6) \sim |\epsilon_1\rangle \otimes \cdots \otimes |\epsilon_6\rangle \in (\mathbb{C}^2)^6$$

We also use a notation where we list the slots with entry -1 in square brackets, so

$$[\emptyset] = |++++++\rangle$$
$$[3456] = |++----\rangle$$
$$[145] = |-++--+\rangle$$

Then a N=1 superconformal generator is given by

$$\Psi := [\emptyset] + i[123456] + ([1234] + [3456] + [1256]) + i([12] + [56] + [34]) \\ + ([135] + [245] + [236] + [146]) - i([246] + [235] + [136] + [145])$$

Where does this complicated looking thing come from and why does it have this form? The answer involves error correcting codes, both classical and quantum.

The hexacode $\mathcal{H}_6$ is a classical error correcting code, a rank 3 subspace of $(\mathbb{F}_4)^6$. It can be specified in several ways, perhaps simplest is to just give generators:

$$b_1 = (1, 0, 0, 1, \bar{\omega}, \omega)$$
$$b_2 = (0, 1, 0, 1, \omega, \bar{\omega})$$
$$b_3 = (0, 0, 1, 1, 1, 1)$$

We also need a group homomorphism from $\mathbb{F}_4^+$, which is $\mathbb{F}_4$ considered as an Abelian group with + group law, to the quaternion subgroup of SU(2).

This extends in an obvious way to a map from the hexacode to SU(2)^6 and one finds the surprising result that

$$h(w_1)h(w_2) = h(w_1 + w_2), \quad w_1, w_2 \in \mathcal{H}_6$$

$$h(\underline{0}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$h(\underline{1}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2$$
$$h(\underline{\omega}) = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = i\sigma^1$$
$$h(\underline{\bar{\omega}}) = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} = -i\sigma^3 .$$

It follows from this and properties of the hexacode that

$$P = \frac{1}{64} \sum_{w \in \mathcal{H}_6} h(w)$$

is a rank one projection operator and one checks that the N=1 superconformal generator is given by

$$\Psi = 16 P | + + + + + + \rangle$$

We are using the earlier correspondence between dimension 3/2=6x1/4 operators and vectors in $(\mathbb{C}^2)^6$. In the language of quantum computation, Ψ is a state in a 6-qubit system, and one with very special properties. It is a maximally entangled state and closely related to the smallest quantum error correcting code capable of correcting an arbitrary one qubit error.

Entangled quantum states carry information and can be used to encode quantum information in a way that protects it again errors. The smallest code that can detect and correct a single qubit error without destroying quantum information works by embedding a single qubit state into a 5 qubit state:

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|0_L\rangle + \beta|1_L\rangle$$

where

$$\begin{aligned}|0_L\rangle = \frac{1}{4}(&|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle \\ &+ |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \\ &- |11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle \\ &- |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle)\end{aligned}$$

$$|1_L\rangle = X^{\otimes 5}|0_L\rangle$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

If we add one more qubit we can construct a maximally entangled 6 qubit state in that

$$\Psi_{[[6,0,4]]} = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0_L\rangle + |1\rangle \otimes |1_L\rangle\right)$$

the trace over any three qubits gives the density matrix $\rho^{(3)} = \frac{I}{8}$

It turns out that this maximally entangled state, and the state describing the N=1 superconformal generator in the GTVW model are unitarily equivalent (+,-) -> (0,1) (T. Maniero).

$$\Psi_{[[6,0,4]]} = (1 \otimes U_1 \otimes 1 \otimes U_2 \otimes U_2 \otimes 1)\Psi$$

$$U_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \qquad U_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}$$

I have glossed over many interesting details:

1. **The error correcting property is linked to the cancellation of certain terms in the OPE of general dimension 3/2 operators required to have superconformal symmetry.**
2. **The symmetry group preserving the superconformal symmetry is linked to the holomorph of the hexacode, Holomorph(G)=** $G : \mathrm{Aut}(G)$ **.**

3. The 2^8:M20 symmetry preserving (4,4) superconformal symmetry arises as a subgroup of Hol(G).

A similar construction can be used to construct the superconformal generator in a c=12 theory with Conway symmetry studied by Duncan and constructed out of 24 free fermions. Let γ_i be the fermion zero modes in the Ramond sector. The generate a Clifford algebra. Let

$$\gamma_w = \gamma_1^{w_1} \cdots \gamma_{24}^{w_{24}} \qquad\qquad w = (w_1, \cdots w_{24}) \in (\mathbb{F}_2)^{24}$$

There is a non-trivial cocycle $\gamma_{w_1}\gamma_{w_2} = \epsilon(w_1, w_2)\gamma_{w_1+w_2}$ but it is trivializable when restricted to the Golay code $\mathcal{G} \in (\mathbb{F}_2)^{24}$

Let $\tilde{\gamma}_w$ be its trivialization. Then

$$P = \frac{1}{2^{12}} \sum_{w \in \mathcal{G}} \tilde{\gamma}_w$$

is a rank one projection operator and can be used to construct the superconformal generator, now of dimension 3/2=24/16.

This connection between superconformal generators and codes probably extends to other interesting SCFTs with sporadic symmetry groups such as the c=24 Monster CFT.

I would like to thank Bernard for his far-reaching insights into exceptional structures in physics and mathematics and wish him many more exciting discoveries in the future.

Thank You

$$V_{s^1}(z_1)V_{s^2}(z_2) \sim \frac{\bar{s}^1 s^2}{z_{12}^3} + \kappa_1 \sum_A \frac{\bar{s}^1 \Sigma^A s^2}{z_{12}^2} J^A + \kappa_1^2 \sum_{\alpha<\beta} \frac{\bar{s}^1 \Sigma^{AB} s^2}{z_{12}} J^A J^B + \kappa_2 \frac{\bar{s}^1 s^2}{z_{12}} T(z_0) + + \cdots$$

$$V_s(z_1)V_s(z_2) \sim \frac{\bar{s}s}{z_{12}^3} + \kappa_1^2 \sum_{\alpha<\beta} \frac{\bar{s}\Sigma^{AB} s}{z_{12}} J^A J^B + \frac{1}{2}\frac{\bar{s}s}{z_{12}} T + + \cdots$$

N=1 superconformal OPE implies that

$$\bar{s}\Sigma^{AB} s = 0 \qquad 1 \leq \alpha < \beta \leq 6$$