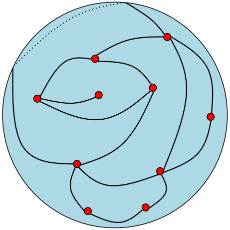
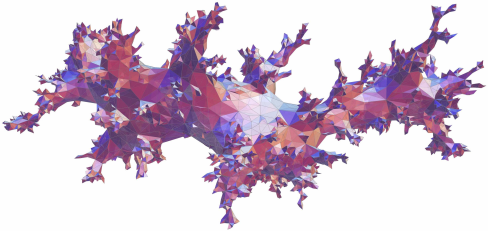
Brownian geometry

Jean-François Le Gall

Université Paris-Sud Orsay

Conference in honour of Bernard Julia





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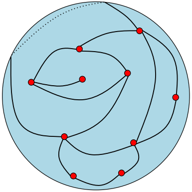

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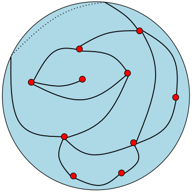

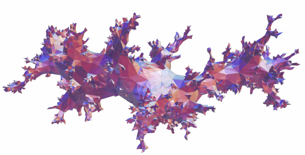

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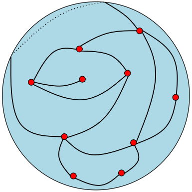

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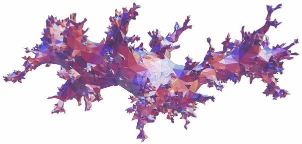

Strong analogy with Brownian motion, which is a canonical model for a random curve in space, obtained as the scaling limit of random walks on the lattice.

1. Statement of the main result

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A planar map is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).

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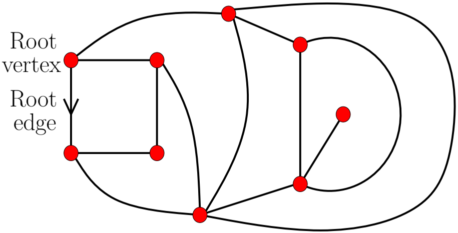

A rooted quadrangulation with 7 faces

Faces = connected components of the complement of edges

p-angulation:

- each face is bounded by p edges

$p = 3$: triangulation
$p = 4$: quadrangulation

Rooted map: distinguished oriented edge

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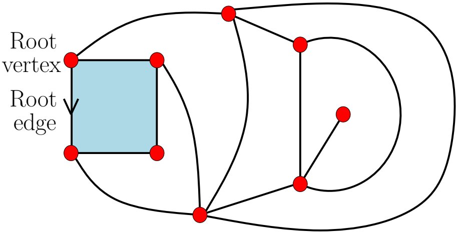

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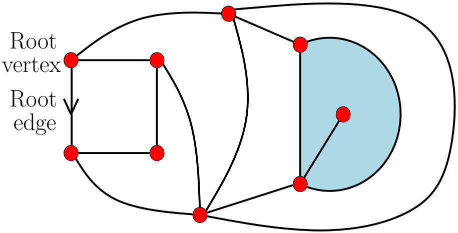

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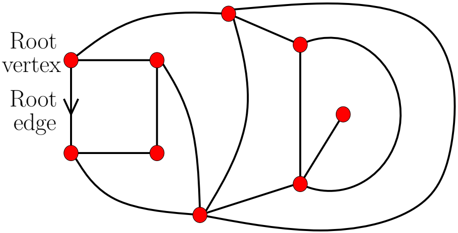

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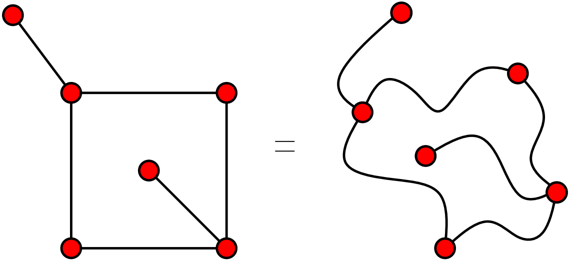

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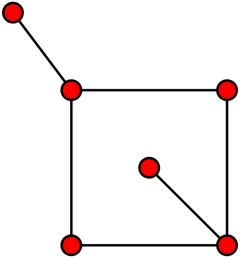

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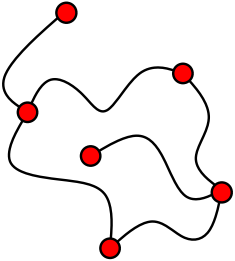
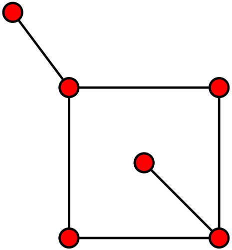


Two different planar maps:



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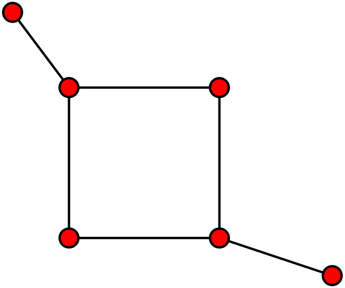
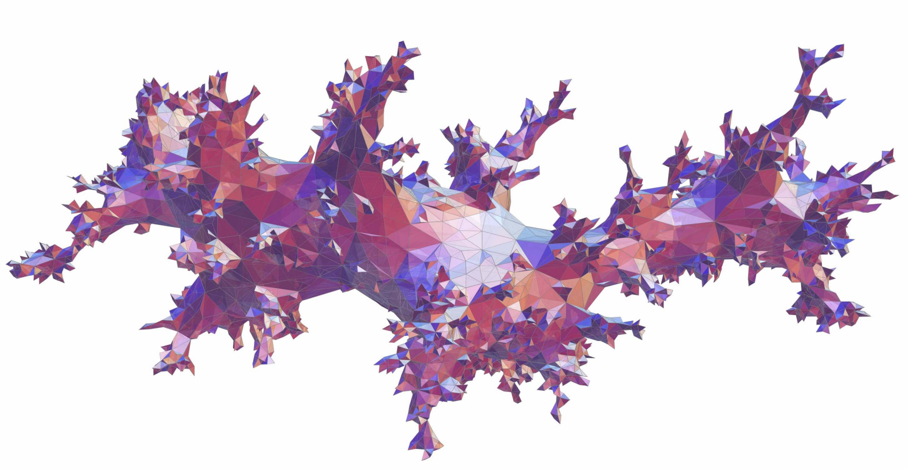


A large triangulation of the sphere (simulation: N. Curien)

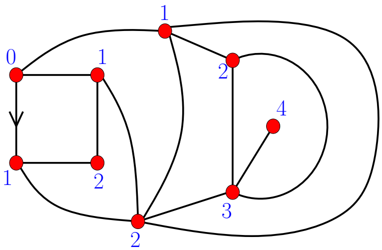
Can we get a continuous model out of this ?



Planar maps as metric spaces

M planar map

- $V(M)$ = set of vertices of M
- d_{gr} graph distance on $V(M)$
- $(V(M), d_{\mathrm{gr}})$ is a (finite) metric space



In blue : distances from the root vertex

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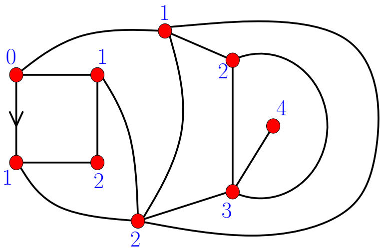

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$\mathbb{M}_n^p = \{$rooted $p-$angulations with n faces$\}$

$\mathbb{M}_n^p$ is a finite set (*finite number of possible "shapes"*)

Choose M_n uniformly at random in $\mathbb{M}_n^p$.

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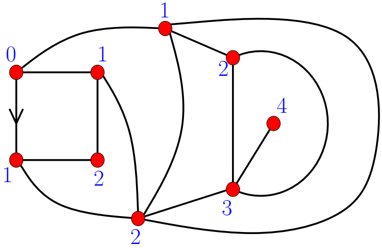

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View $(V(M_n), d_{\mathrm{gr}})$ as a random variable with values in

$$\mathbb{K} = \{\text{compact metric spaces, modulo isometries}\}$$

which is equipped with the Gromov-Hausdorff distance.

The Gromov-Hausdorff distance

The Hausdorff distance. K_1, K_2 compact subsets of a metric space

$$d_{\mathrm{Haus}}(K_1, K_2) = \inf\{\varepsilon > 0 : K_1 \subset U_\varepsilon(K_2) \text{ and } K_2 \subset U_\varepsilon(K_1)\}$$

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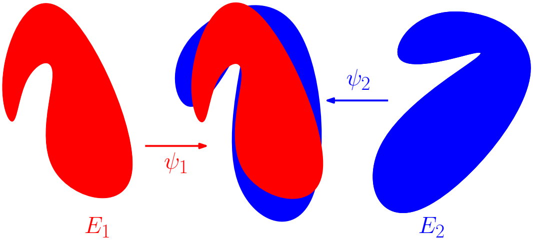
Definition (Gromov-Hausdorff distance)

If (E_1,d_1) and (E_2,d_2) are two compact metric spaces,

$$d_{\mathrm{GH}}(E_1,E_2) = \inf\{d_{\mathrm{Haus}}(\psi_1(E_1),\psi_2(E_2))\}$$

the infimum is over all isometric embeddings $\psi_1 : E_1 \to E$ and $\psi_2 : E_2 \to E$ of E_1 and E_2 into the same metric space E.



Gromov-Hausdorff convergence of rescaled maps

Fact

If $\mathbb{K} = \{$*isometry classes of compact metric spaces*$\}$, *then*

$$(\mathbb{K}, d_{\mathrm{GH}}) \text{ is a separable complete metric space (Polish space)}$$

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$\to$ If M_n is uniformly distributed over $\{p-$ angulations with n faces$\}$, it makes sense to study the convergence in distribution of

$$(V(M_n), n^{-a} d_{\mathrm{gr}})$$

as random variables with values in $\mathbb{K}$.

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Choice of the rescaling factor n^{-a} : $a > 0$ is chosen so that $\mathrm{diam}(V(M_n)) \approx n^a$.

$\Rightarrow a = \frac{1}{4}$ [cf Chassaing-Schaeffer PTRF 2004 for quadrangulations]

Main result: The Brownian sphere

$\mathbb{M}_n^p = \{\text{rooted } p-\text{angulations with } n \text{ faces}\}$

M_n uniform over $\mathbb{M}_n^p$, $V(M_n)$ vertex set of M_n, $d_{\rm gr}$ graph distance

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Theorem (LG, Miermont for $p = 4$)

Suppose that either $p = 3$ (triangulations) or $p \geq 4$ is even. Set

$$c_3 = 6^{1/4} \quad , \quad c_p = \left(\frac{9}{p(p-2)}\right)^{1/4} \quad \textit{if } p \textit{ is even.}$$

Then,

$$(V(M_n), c_p\, n^{-1/4}\, d_{\rm gr}) \xrightarrow[n\to\infty]{\rm (d)} (\mathbf{m}_\infty, D^*)$$

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Remarks. Extensions to many other random planar maps (Abraham, Addario-Berry, Albenque, Miermont, Bettinelli, etc.)

Expect the result to be also true for any odd value of p

Two properties of the Brownian sphere

Theorem (Hausdorff dimension)

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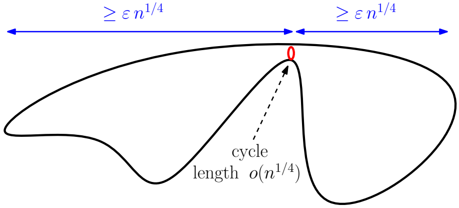
Consequence: for a typical planar map M_n with n faces, diameter $\approx n^{1/4}$ but: no cycle of size $o(n^{1/4})$ in M_n, such that both sides have diameter $\geq \varepsilon n^{1/4}$



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- probability theory: models for a Brownian surface
  - analogy with Brownian motion as continuous limit of discrete paths
  - universality of the limit
  - asymptotic properties of "typical" large planar maps

Recent progress

Miller, Sheffield (2015-2016) have developed a program aiming to relate the Brownian sphere with Liouville quantum gravity:

- new construction of the Brownian sphere via the Gaussian free field and the random growth process called Quantum Loewner Evolution (an analog of the celebrated SLE processes studied by Lawler, Schramm and Werner)
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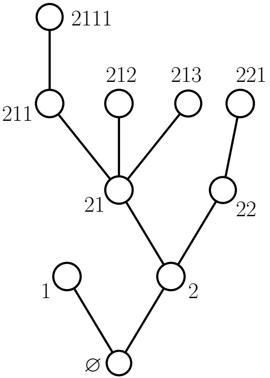
More recently, direct construction of the Liouville quantum gravity metric (via the exponential of the Gaussian free field multiplied by a parameter γ): Ding, Dubedat, Gwynne-Miller (2019), ... A special case ($\gamma = \sqrt{8/3}$) should give the Brownian sphere metric. Other values of γ are expected to correspond to planar maps weighted by a statistical physics model.

2. A key tool: Bijections between maps and trees



A plane tree τ with vertex set
$V(\tau) = \{\varnothing, 1, 2, 21, 22, 212, \ldots\}$
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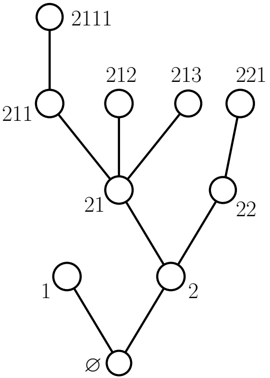

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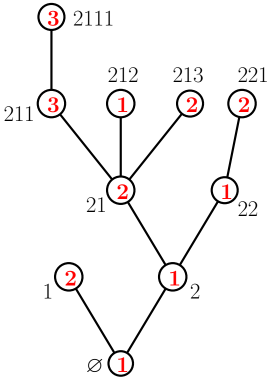

A well-labeled tree $(\tau, (\ell_v)_{v \in V(\tau)})$
Properties of labels:

- $\ell_\varnothing = 1$
- $\ell_v \in \{1, 2, 3, \ldots\}, \forall v$
- $|\ell_v - \ell_{v'}| \leq 1$, if v, v' neighbors

Coding maps with trees, the case of quadrangulations

$\mathbb{T}_n = \{\text{well-labeled trees with } n \text{ edges}\}$
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There is a bijection $\Phi : \mathbb{T}_n \longrightarrow \mathbb{M}_n^4$ such that, if $M = \Phi(\tau, (\ell_v)_{v \in V(\tau)})$, then

$$V(M) = V(\tau) \cup \{\partial\} \qquad (\partial \text{ is the root vertex of } M)$$
$$d_{\mathrm{gr}}(\partial, v) = \ell_v \quad , \forall v \in V(\tau)$$

Key properties.

- Vertices of τ become vertices of M
- The label in the tree becomes the distance from the root in the map.

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Coding of more general maps: Bouttier, Di Francesco, Guitter (2004)

Schaeffer's bijection between quadrangulations and well-labeled trees

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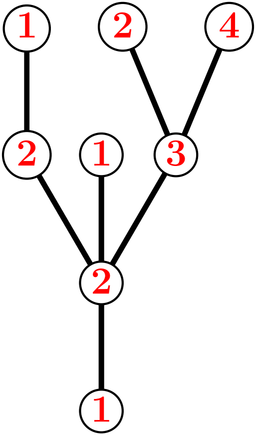

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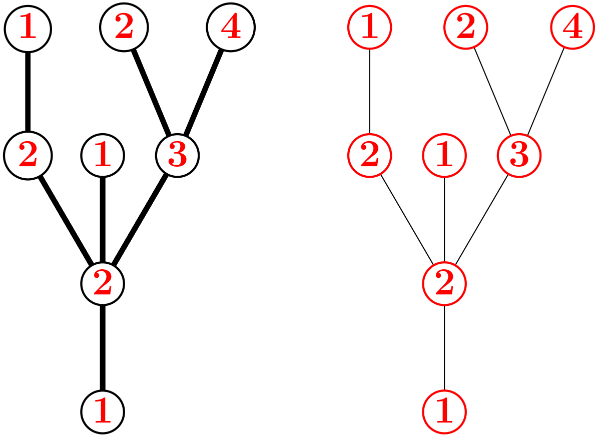

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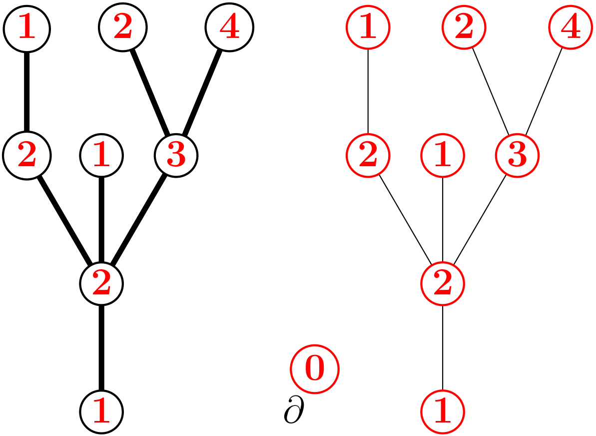

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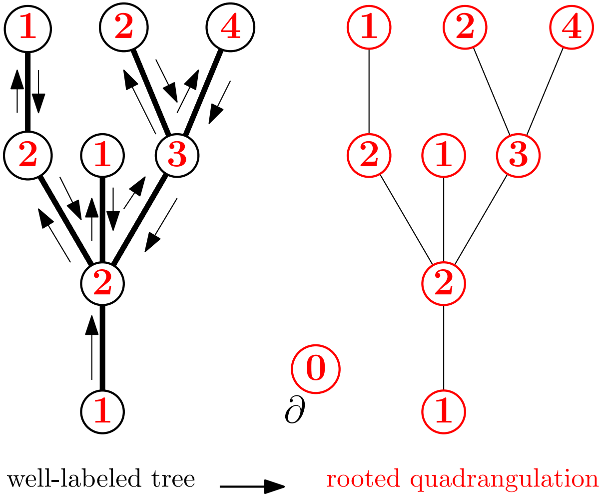

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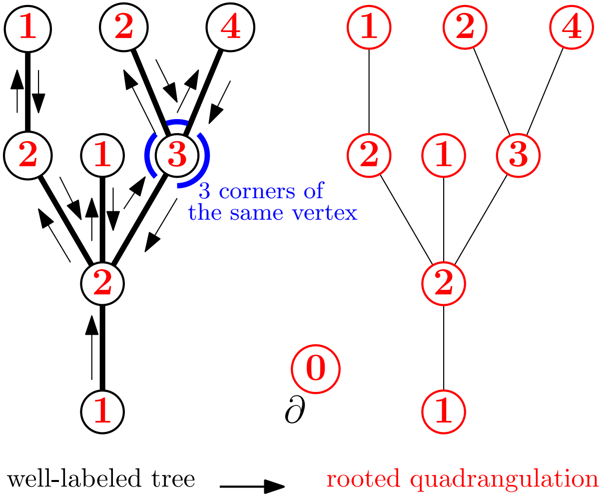
- Add extra vertex ∂ labeled 0
- Follow the contour of the tree

Schaeffer's bijection between quadrangulations and well-labeled trees



Rules

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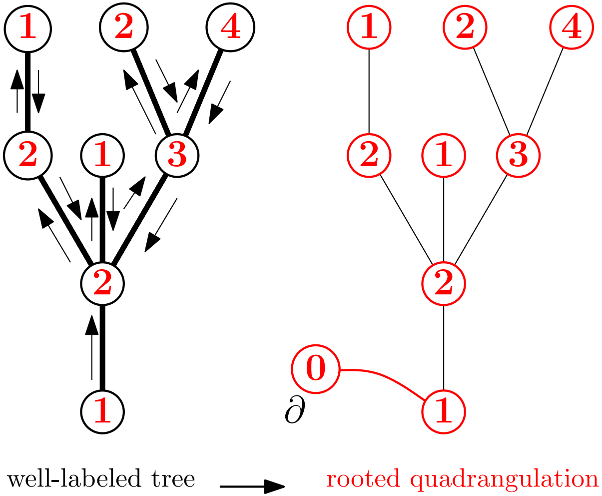

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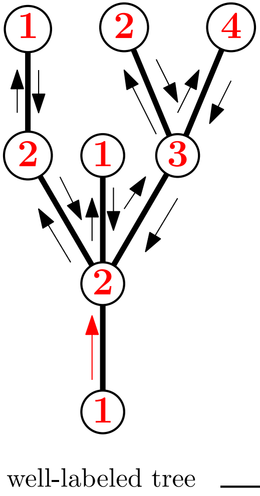

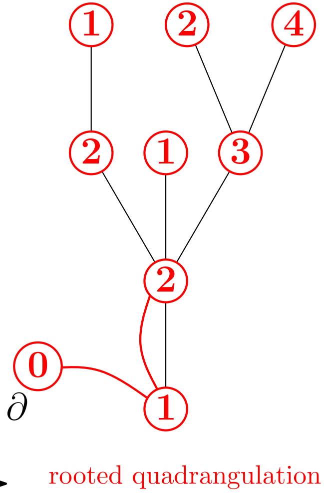

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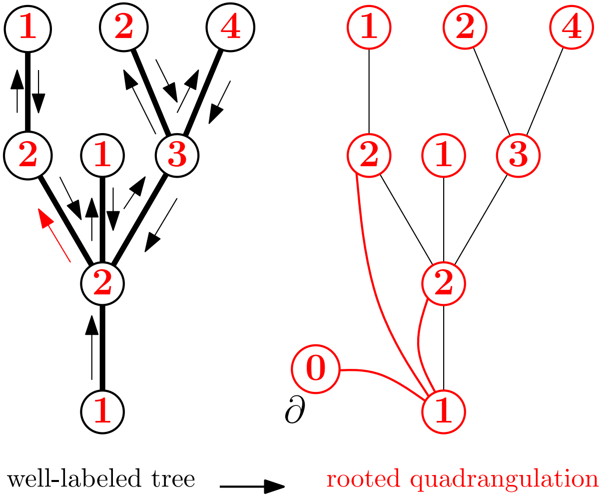

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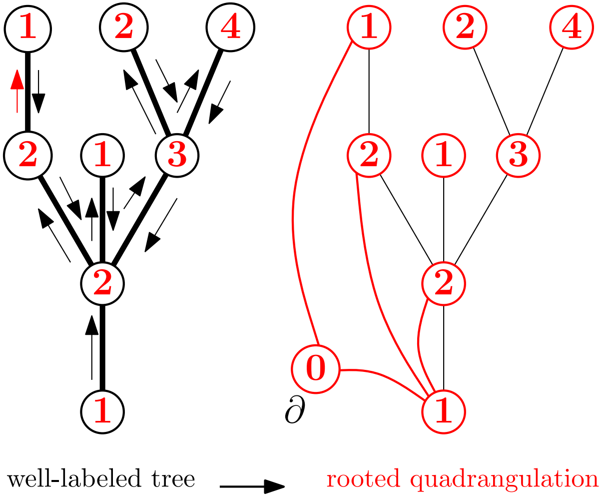

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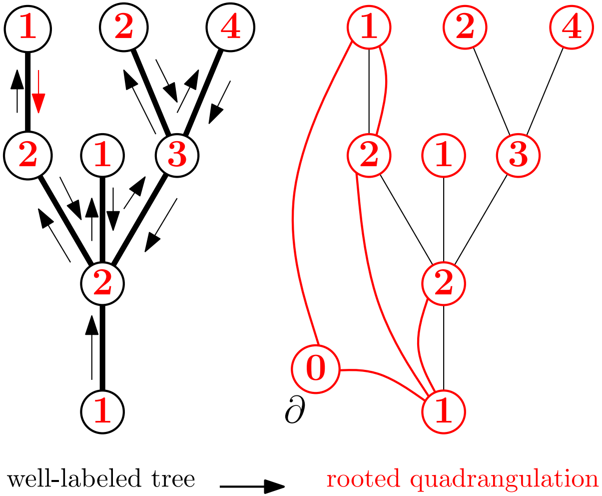

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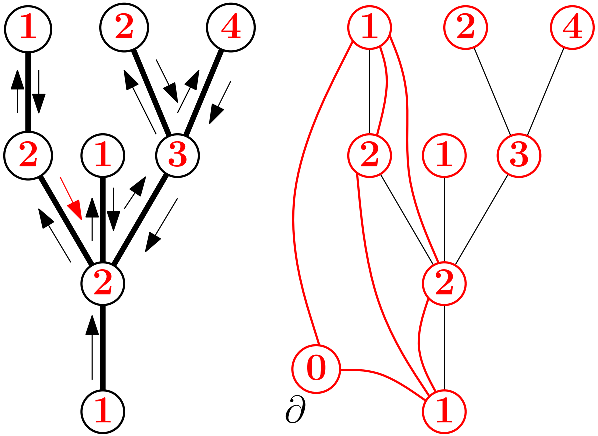

well-labeled tree $\longrightarrow$ rooted quadrangulation

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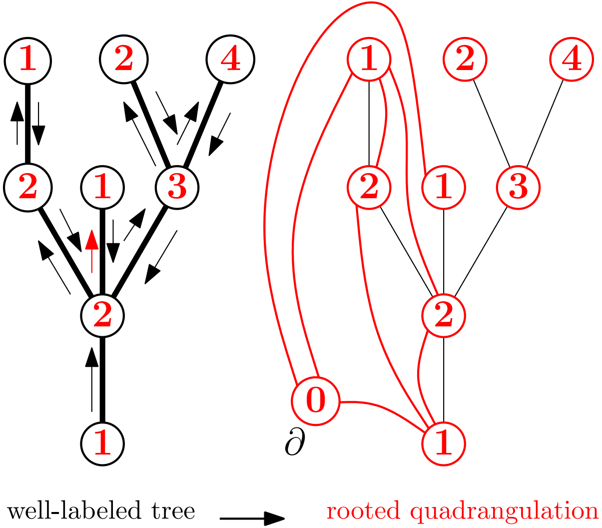

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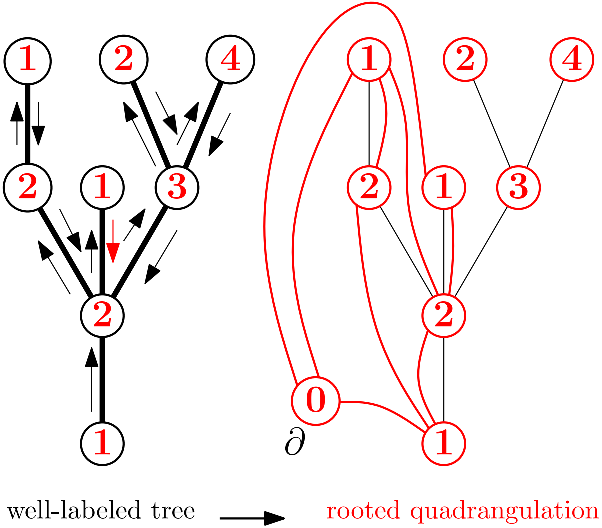

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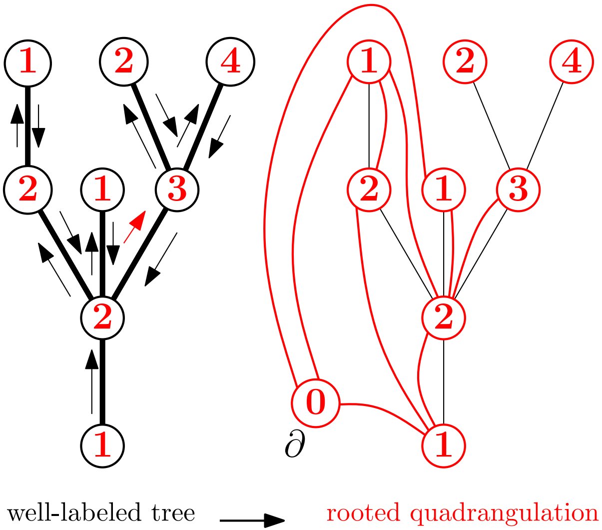

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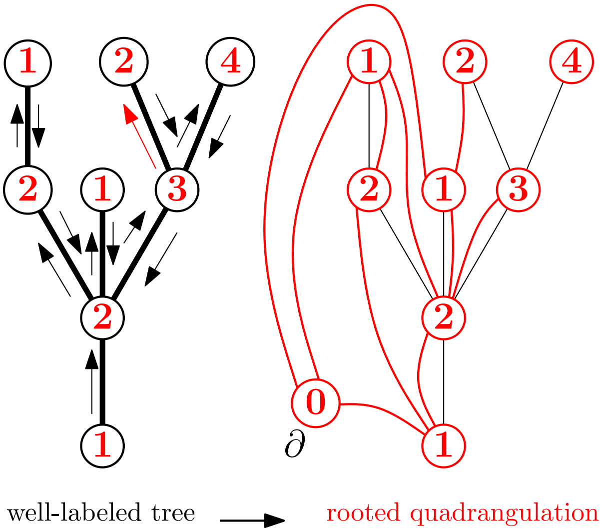

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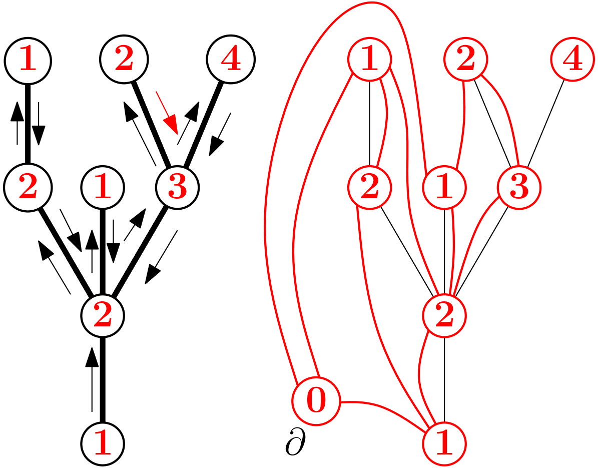

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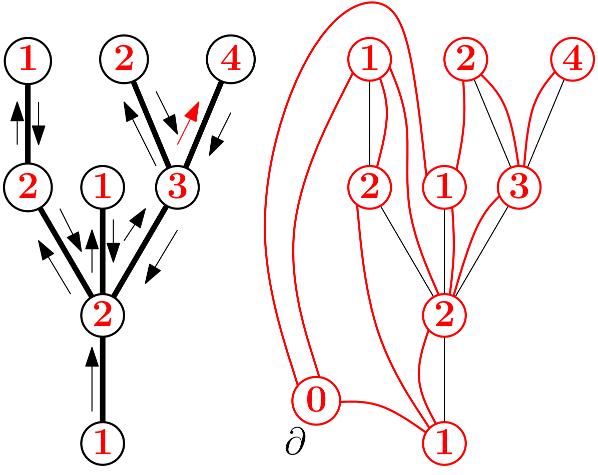

well-labeled tree → rotated quadrangulation

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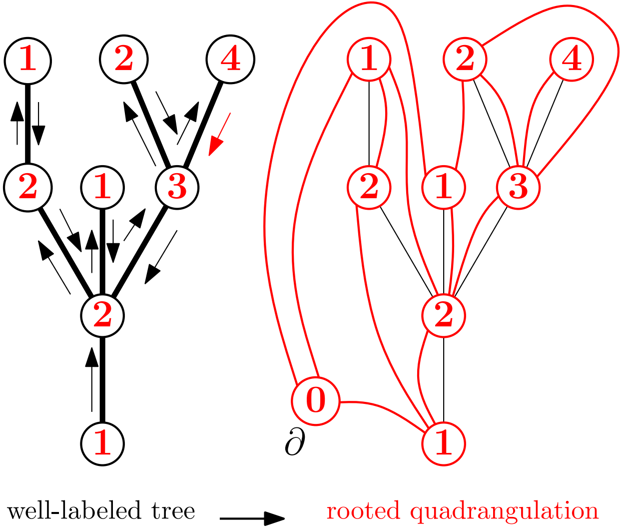

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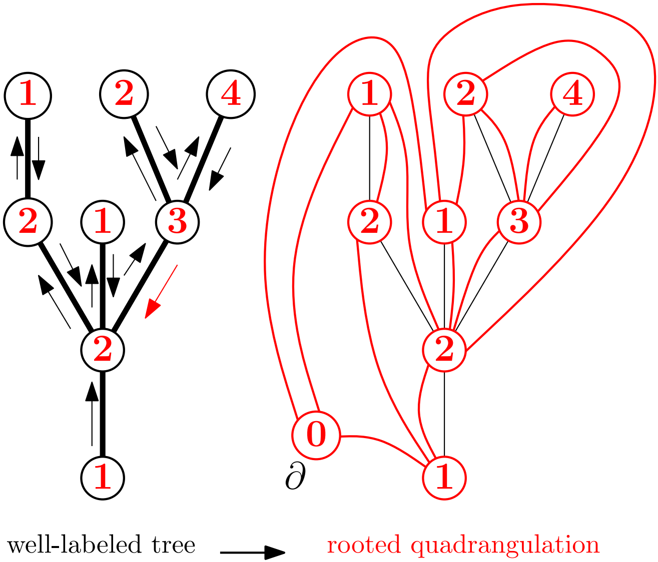

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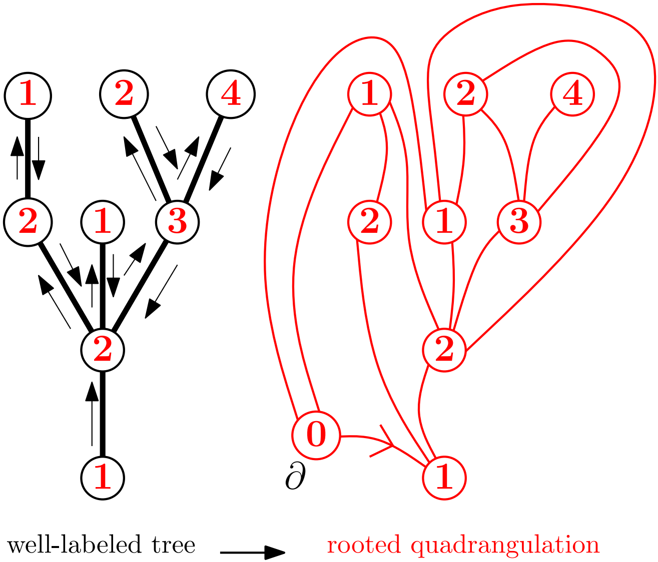

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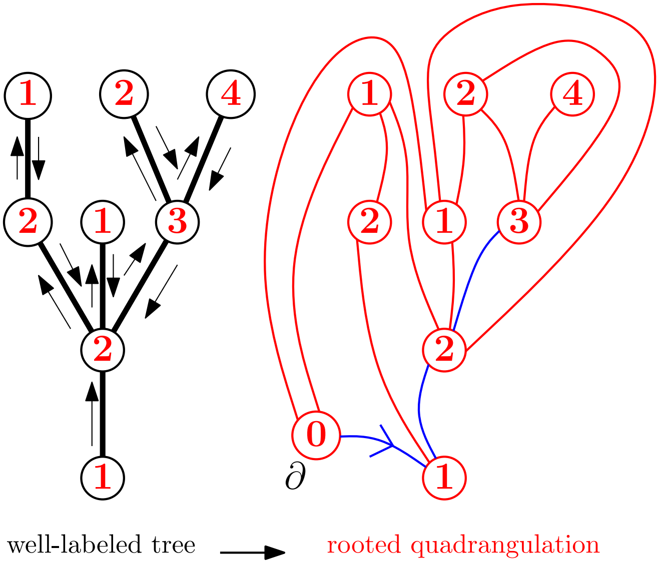

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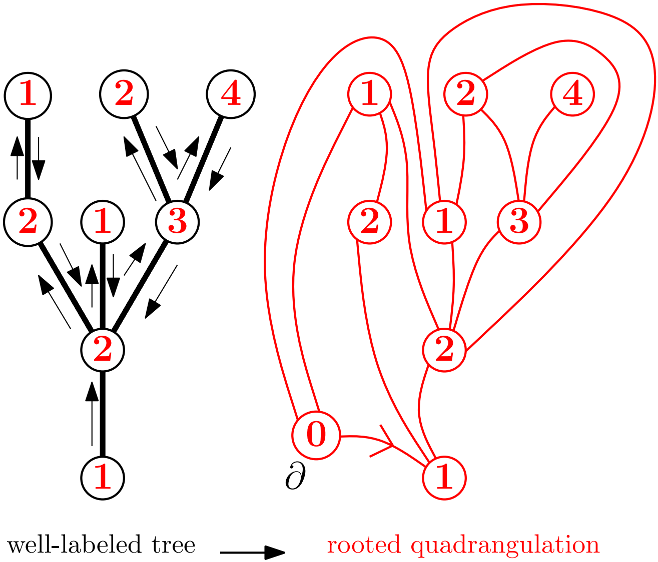

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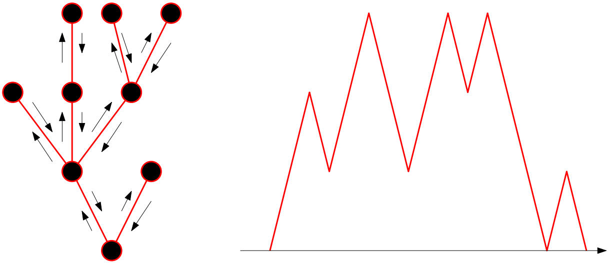
Coding a (discrete) plane tree by its contour function (or Dyck path):



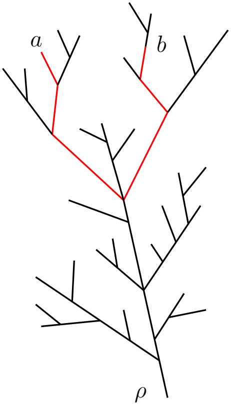
The notion of a real tree

Definition

A real tree, or $\mathbb{R}$-tree, is a (compact) metric space $\mathcal{T}$ such that:

- any two points $a, b \in \mathcal{T}$ are joined by a unique continuous and injective path (up to re-parametrization)
- this path is isometric to a line segment

$\mathcal{T}$ is a rooted real tree if there is a distinguished point ρ, called the root.



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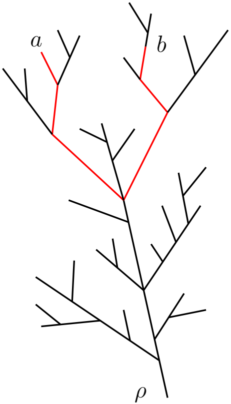

Remark. A real tree can have

- infinitely many branching points
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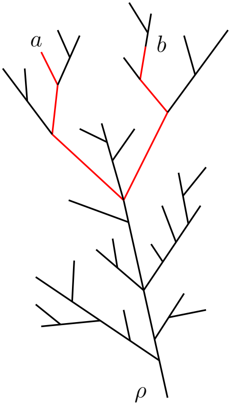

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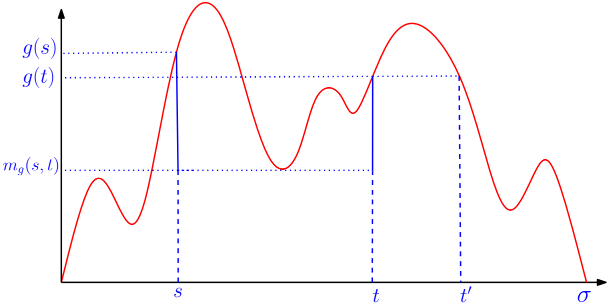
Fact. The coding of discrete trees by contour functions can be extended to real trees: also gives a cyclic ordering on the tree.

The real tree coded by a function g

$g : [0, 1] \longrightarrow [0, \infty)$
continuous,

$g(0) = g(1) = 0$

$m_g(s, t) = \min_{[s \wedge t, s \vee t]} g$



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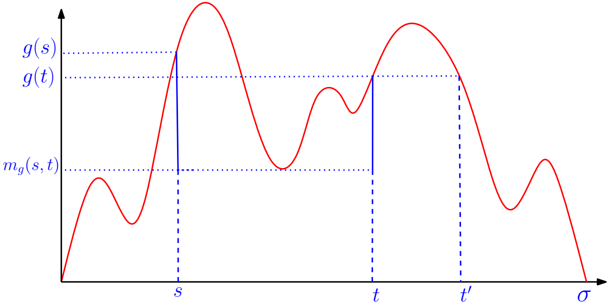

$d_g(s,t) = g(s) + g(t) - 2\, m_g(s,t)$ pseudo-metric on $[0,1]$

$t \sim t'$ iff $d_g(t,t') = 0$ (or equivalently $g(t) = g(t') = m_g(t,t')$)

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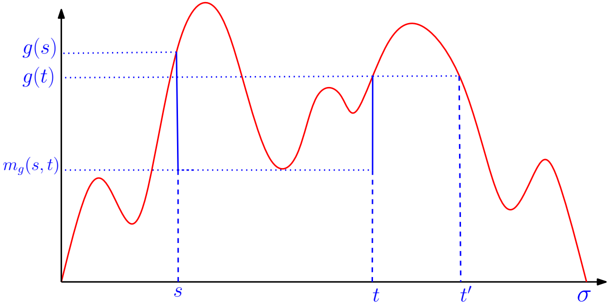

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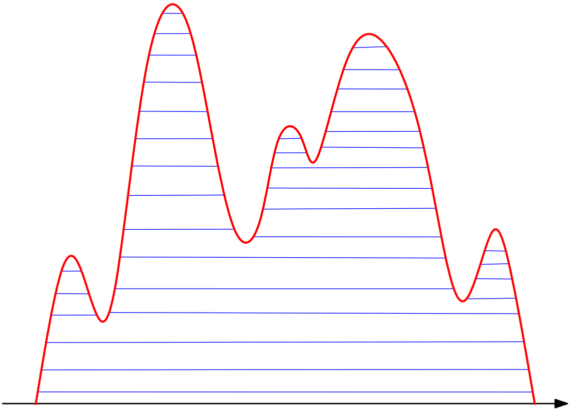
Proposition

$\mathcal{T}_g := [0,1]/\sim$ equipped with d_g is a real tree, called the tree coded by g. It is rooted at $\rho = 0$.

The canonical projection $[0,1] \to \mathcal{T}_g$ induces a cyclic ordering on $\mathcal{T}_g$

Coding a tree by a function



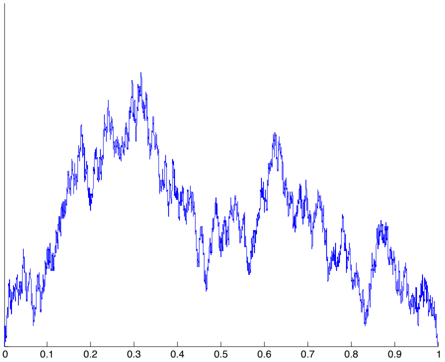
Every horizontal blue line segment below the curve is identified to a single point.

Definition of the CRT

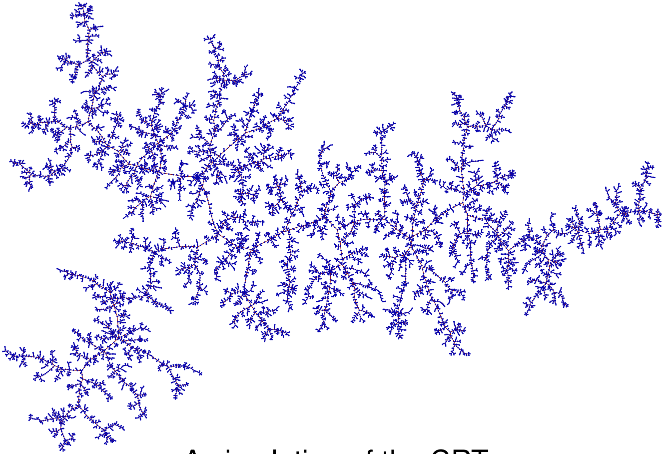
Let $(\mathbf{e}_t)_{0\leq t\leq 1}$ be a Brownian excursion with duration 1 (= Brownian motion started from 0 conditioned to be at 0 at time 1 and to stay ≥ 0)

Definition

The CRT $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the (random) real tree coded by the Brownian excursion $\mathbf{e}$.



Simulation of a Brownian excursion



A simulation of the CRT

(simulation: I. Kortchemski)

Assigning Brownian labels to a real tree

Let $(\mathcal{T}, d)$ be a real tree with root ρ.

$(Z_a)_{a\in\mathcal{T}}$: Brownian motion indexed by $(\mathcal{T}, d)$
= centered Gaussian process such that

- $Z_\rho = 0$
- $E[(Z_a - Z_b)^2] = d(a, b), \qquad a, b \in \mathcal{T}$

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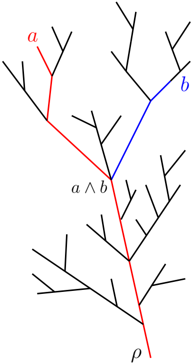

Labels evolve like Brownian motion along the branches of the tree:

- The label Z_a is the value at time $d(\rho, a)$ of a standard Brownian motion
- Similar property for Z_b, but one uses
  - the same BM between 0 and $d(\rho, a \wedge b)$
  - an independent BM between $d(\rho, a \wedge b)$ and $d(\rho, b)$

The definition of the Brownian sphere

$(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the CRT, $(Z_a)_{a\in\mathcal{T}_{\mathbf{e}}}$ Brownian motion indexed by the CRT (Two levels of randomness!).

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Set, for every $a, b \in \mathcal{T}_{\mathbf{e}}$,

$$D^0(a,b) = Z_a + Z_b - 2\max\Big(\min_{c\in[a,b]} Z_c, \min_{c\in[b,a]} Z_c\Big)$$

where $[a, b]$ is the "interval" from a to b corresponding to the cyclic ordering on $\mathcal{T}_{\mathbf{e}}$ (vertices visited when going from a to b in clockwise order around the tree).

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Then set

$$D^*(a,b) = \inf_{a_0=a,a_1,\ldots,a_{k-1},a_k=b} \sum_{i=1}^{k} D^0(a_{i-1},a_i),$$

$$a \approx b \text{ if and only if } D^*(a,b)=0 \quad \text{(equivalent to } D^0(a,b)=0).$$

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Definition

The Brownian sphere $\mathbf{m}_\infty$ is the quotient space $\mathbf{m}_\infty := \mathcal{T}_{\mathbf{e}}/\approx$, which is equipped with the distance induced by D^*.

Summary and interpretation

Starting from the CRT $\mathcal{T}_{\mathbf{e}}$, with Brownian labels $Z_a, a \in \mathcal{T}_{\mathbf{e}}$,
$\to$ Identify two vertices $a, b \in \mathcal{T}_{\mathbf{e}}$ if:

- they have the same label $Z_a = Z_b$,
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Remark. Not many vertices are identified:

- A "typical" equivalence class is a singleton.
- Equivalence classes may contain at most 3 points.

Still these identifications drastically change the topology.

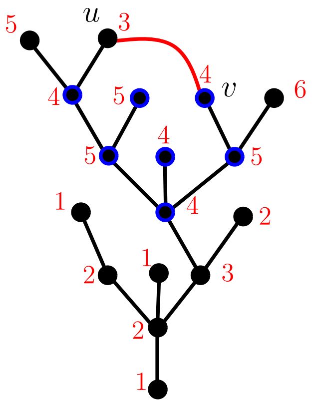
Interpretation of the equivalence relation $\approx$

In Schaeffer's bijection:
$\exists$ edge between u and v if

- $\ell_u = \ell_v - 1$
- $\ell_w \geq \ell_v$, $\forall w \in]u, v]$

Explains why in the continuous limit

$$Z_a = Z_b = \min_{c \in [a,b]} Z_c$$
$$\Rightarrow a \text{ and } b \text{ are identified}$$



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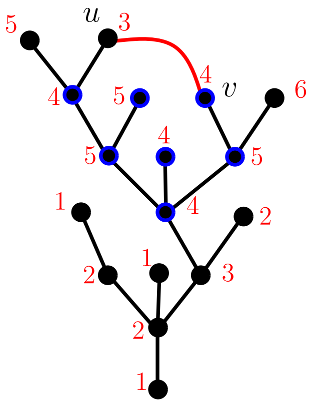

Key points of the proof of the main theorem:

- Prove the converse (no other pair of points are identified)
- Obtain the formula for the limiting distance D^*

Properties of distances in the Brownian sphere

Let ρ_* be the (unique) vertex of $\mathcal{T}_{\mathbf{e}}$ such that

$$Z_{\rho_*} = \min_{c \in \mathcal{T}_{\mathbf{e}}} Z_c$$

Then, for every $a \in \mathcal{T}_{\mathbf{e}}$,

$$D^*(\rho_*, a) = Z_a - \min Z.$$

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("follows" from the analogous property in the discrete setting)

No such simple expression for $D^*(a, b)$ in terms of labels, but

$$D^*(a,b) \leq D^0(a,b) = Z_a + Z_b - 2\max\Big(\min_{c\in[a,b]} Z_c, \min_{c\in[b,a]} Z_c\Big)$$

(in the discrete setting, corresponds to constructing a path between a and b from the union of the two geodesics from a, resp. from b, to ∂ until the point when they merge)

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Then, for every $a \in \mathcal{T}_{\mathbf{e}}$,

$$D^*(\rho_*, a) = Z_a - \min Z.$$

("follows" from the analogous property in the discrete setting)

No such simple expression for $D^*(a, b)$ in terms of labels, but

$$D^*(a, b) \leq D^0(a, b) = Z_a + Z_b - 2 \max \Big(\min_{c \in [a,b]} Z_c, \min_{c \in [b,a]} Z_c \Big)$$

(in the discrete setting, corresponds to constructing a path between a and b from the union of the two geodesics from a, resp. from b, to ∂ until the point when they merge)

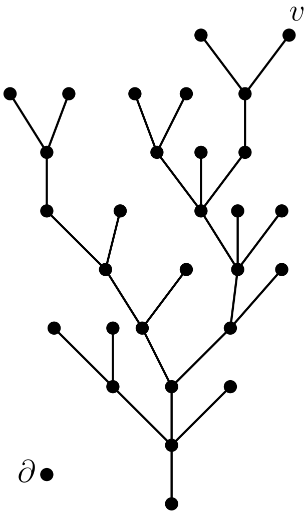
D^* is the maximal metric that satisfies this inequality

4. Geodesics in the Brownian sphere

Geodesics in quadrangulations

Use Schaeffer's bijection between quadrangulations and well-labeled trees.

To construct a geodesic from *v* to ∂:



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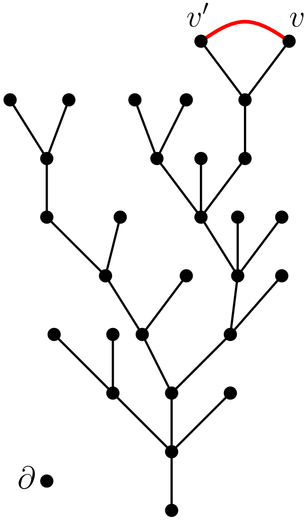

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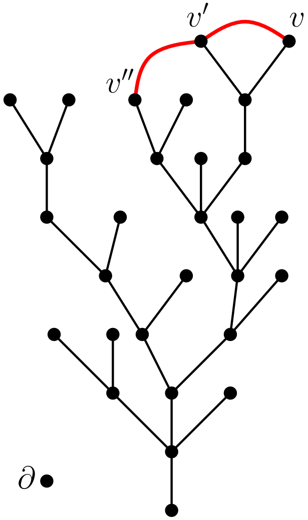

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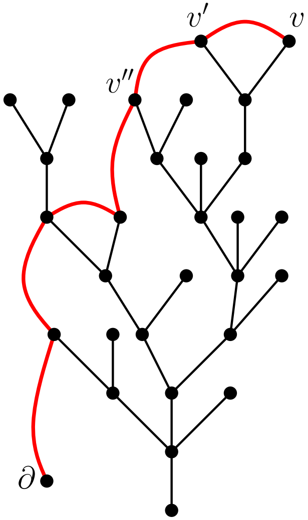
- And so on.
- Eventually one reaches ∂.



Geodesics to ρ_* in the Brownian sphere

Recall : ρ_* is the unique point of $\mathcal{T}_{\mathbf{e}}$ s.t.

$$Z_{\rho_*} = \min_{c \in \mathcal{T}_{\mathbf{e}}} Z_c$$

then, for every $b \in \mathcal{T}_{\mathbf{e}}$,

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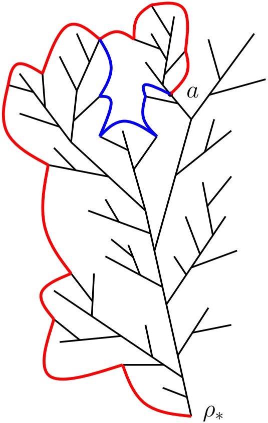
$\varphi_a(t) =$ last vertex b before a s.t. $\widetilde{Z}_b = t$

("last", "before" refer to the cyclic order)



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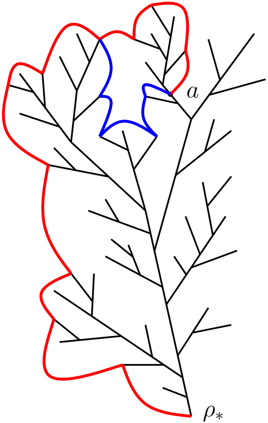

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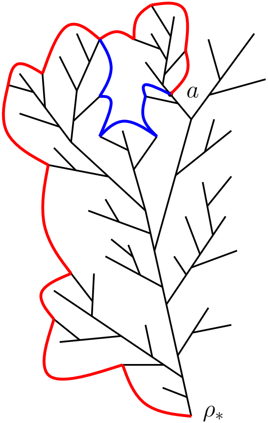

Fact

All geodesics to ρ_* are of this form.

If a is not a leaf, there are several possible choices, depending on which side of a one starts.

The main result about geodesics

Define the skeleton of $\mathcal{T}_{\mathbf{e}}$ by $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}}) = \mathcal{T}_{\mathbf{e}} \backslash \{\text{leaves of } \mathcal{T}_{\mathbf{e}}\}$ and set

$\mathrm{Skel} = \pi(\mathrm{Sk}(\mathcal{T}_{\mathbf{e}}))$, where $\pi : \mathcal{T}_{\mathbf{e}} \to \mathcal{T}_{\mathbf{e}}/\!\approx\; = \mathbf{m}_\infty$ canonical projection

Then

- the restriction of π to $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}})$ is a homeomorphism onto Skel
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Let $x \in \mathbf{m}_\infty$. Then,

- *if $x \notin \mathrm{Skel}$, there is a unique geodesic from ρ_* to x*
- *if $x \in \mathrm{Skel}$, the number of distinct geodesics from ρ_* to x is the multiplicity $m(x)$ of x in Skel (note: $m(x) \leq 3$).*

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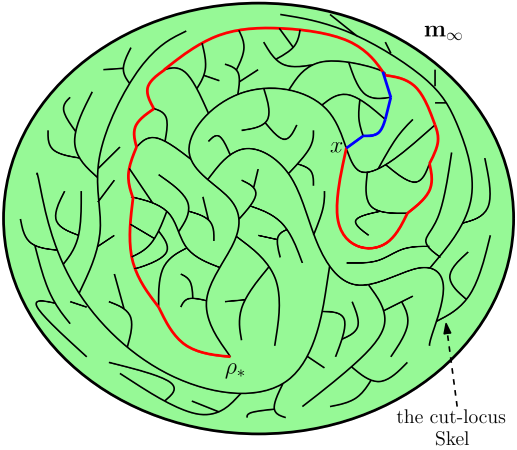
Remarks

- Skel is the cut-locus of $\mathbf{m}_\infty$ relative to ρ_*: cf classical Riemannian geometry [Poincaré, Myers, ...], where the cut-locus is a tree.
- same results if ρ_* replaced by a point chosen "at random" in $\mathbf{m}_\infty$.

Illustration of the cut-locus



The cut-locus Skel
is homeomorphic to
a non-compact real tree
and is dense in $\mathbf{m}_\infty$

Geodesics to ρ_*
do not visit Skel
(except possibly
at their starting point)
but "move around" Skel.

Confluence property of geodesics

Fact: Two geodesics to ρ_* coincide near ρ_*.
(easy from the definition)

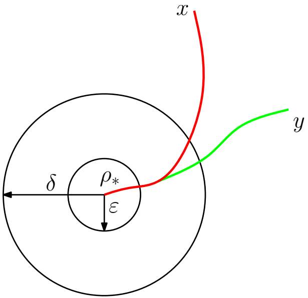
Corollary

Given $\delta > 0$, there exists $\varepsilon > 0$ s.t.

- *if $D^*(\rho_*, x) \geq \delta$, $D^*(\rho_*, y) \geq \delta$*
- *if γ is any geodesic from ρ_* to x*
- *if γ' is any geodesic from ρ_* to y*

then

$$\gamma(t) = \gamma'(t) \quad \textit{for all } t \leq \varepsilon$$



"Only one way" of leaving ρ_* along a geodesic.
(also true if ρ_* is replaced by a typical point of $\mathbf{m}_\infty$)

[This property is also crucial when studying the metric defined from the exponential of the Gaussian free field]

Uniqueness of geodesics in discrete maps

M_n uniform distributed over $\mathbb{M}_n^p = \{p-\text{angulations with } n \text{ faces}\}$
$V(M_n)$ set of vertices of M_n, ∂ root vertex of M_n, $d_{\rm gr}$ graph distance

For $v \in V(M_n)$, set $\mathrm{Geo}(\partial \to v) = \{\text{geodesics from } \partial \text{ to } v\}$

If γ, γ' are two discrete paths in M_n (with the same length)

$$d(\gamma, \gamma') = \max_i d_{\rm gr}(\gamma(i), \gamma'(i))$$

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Corollary

Let $\delta > 0$. Then,

$$\frac{1}{n}\#\{v \in V(M_n) : \exists \gamma,\gamma' \in \mathrm{Geo}(\partial \to v),\ d(\gamma,\gamma') \geq \delta n^{1/4}\} \underset{n\to\infty}{\longrightarrow} 0$$

Two discrete geodesics (between two typical points) are within a distance $o(n^{-1/4})$
(Macroscopic uniqueness, also true for "approximate geodesics"= paths with length $d_{\rm gr}(\partial, v) + o(n^{1/4})$)

Conclusions

- The Brownian sphere (and its variants the Brownian disk, the Brownian plane) provide universal models of two-dimensional random geometry.
- The Liouville theory approach (measuring lengths with weights given by the exponential of the Gaussian free field) should provide alternative constructions of these models.
- There are analogous models in higher genus (Brownian torus, etc.)
- Still many open problems.
- What about higher dimensions ?? (cf. Gurau, Rivasseau,...) The combinatorics become much more difficult.

Thank you for your attention