

From E_7 to E_{10}

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Exceptional Dimensions
A Conference in Honour of Bernard Julia
16-17 December 2019, Institut Henri Poincaré, Paris

THE $N = 8$ SUPERGRAVITY THEORY. I. THE LAGRANGIAN

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Received 25 September 1978

The $SO(8)$ supergravity action is constructed in closed form. A local $SU(8)$ group as well as the exceptional group E_7 are invariances of the equations of motion and of a new first order lagrangian.

Genericity of Cosmological Singularities?

Landau 1959: Is the big bang singularity of Friedmann universes a generic property of general relativistic cosmologies, or is it an artefact of the high degree of symmetry of these solutions?

Khalatnikov and Lifshitz 1963: look for generic **inhomogeneous** and **anisotropic** solutions near a singularity

$$ds^2 = -dt^2 + (a^2 \ell_i \ell_j + b^2 m_i m_j + c^2 n_i n_j) dx^i dx^j$$

single homogeneous Friedmann scale factor $a(t)$ \rightarrow three inhomogeneous scale factors $a(t, \mathbf{x})$, $b(t, \mathbf{x})$, $c(t, \mathbf{x})$

KL63 did not succeed in finding the “general” solution of the complicated, coupled dynamics of a, b, c and tentatively concluded that a singularity is not generic.

Structure of Cosmological Singularities

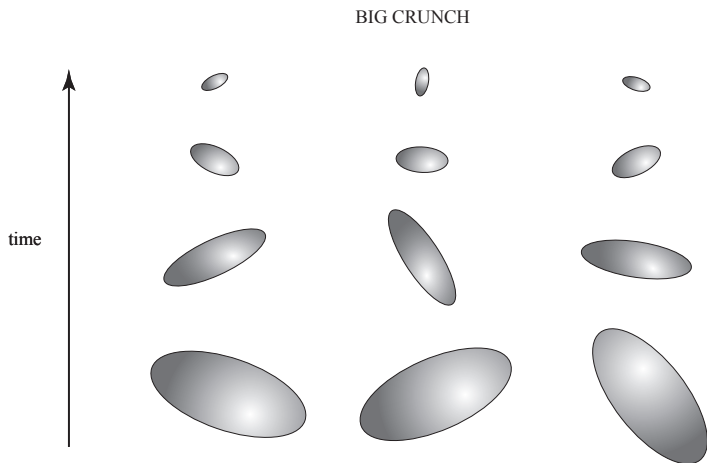
local collapse: Penrose 1965; cosmology: Hawking 1966-7, Hawking-Penrose 1970: Theorems about genericity of cosmological “singularity”. They prove generic “incompleteness” of spacetime, without giving any information about the “singularity”.

Belinsky, Khalatnikov, Lifshitz 1969:

- introduce a new approach to construct the “general” solution near $abc \rightarrow 0$ of the coupled (inhomogeneous) dynamics of $a(t, \mathbf{x})$, $b(t, \mathbf{x})$, $c(t, \mathbf{x})$,
- find that, at each point of space \mathbf{x} , the dynamics of a, b, c is **chaotic**.

The BKL conjecture has been confirmed both by numerical simulations (Weaver-Isenberg-Berger 1998, Berger-Moncrief 1998, Berger et al 1998-2001; Garfinkle 2002-2007; Berger’s Living Review) and by analytical studies (Damour-Henneaux-Nicolai 2003; Ugge et al 2003-2007; Damour-De Buyl 2008).

BKL chaos near a big bang or a big crunch



Dynamics of BKL a, b, c system

January 1968, at the Institut Henri Poincaré, Isaak Khalatnikov gives a seminar in which he announces to the western world the results of BKL. He shows the system of equations for the three local scale factors a, b, c [with new time variable $d\tau = -dt/(abc)$]

$$2 \frac{d^2 \ln a}{d\tau^2} = (b^2 - c^2)^2 - a^4$$

$$2 \frac{d^2 \ln b}{d\tau^2} = (c^2 - a^2)^2 - b^4$$

$$2 \frac{d^2 \ln c}{d\tau^2} = (a^2 - b^2)^2 - c^4$$

J.A. Wheeler was in the audience and immediately pointed out the possibility of a mechanical analogy for this model. He informed his former student Charles Misner (who was independently working on the Bianchi IX dynamics) of the BKL results. In 1969 Misner published a mechanical-like, Lagrangian analysis of the Bianchi IX (a, b, c) system under the catchy name of “mixmaster universe”.

Cosmological Billiards

(Misner 1969a, 1969b [quantum], Chitre 1972, . . . , Damour, Henneaux, Nicolai, Julia \geq 2000, . . . , Belinski-Henneaux 2018)

$$ds^2 = -dt^2 + (a^2 \ell_i \ell_j + b^2 m_i m_j + c^2 n_i n_j) dx^i dx^j$$

exponential parametrisation: $a = e^{-\beta^1}$, $b = e^{-\beta^2}$, $c = e^{-\beta^3}$

Lagrangian ruling the dynamics of the β 's at each spatial point

$$\mathcal{L} = \frac{1}{2} G_{ab} \dot{\beta}^a \dot{\beta}^b - V(\beta)$$

Kinetic metric $G_{ab} \dot{\beta}^a \dot{\beta}^b = \sum_a (\dot{\beta}^a)^2 - \left(\sum_a \dot{\beta}^a \right)^2$ (DeWitt metric)

Potential $V(\beta) = a^4 + b^4 + c^4 - 2(a^2 b^2 + b^2 c^2 + c^2 a^2) = \sum_a c_A(\dots) e^{-2w_A(\beta)}$

Wall forms $w_A(\beta)$: e.g. gravitational walls: $w_{abc}^{(g)}(\beta) = \sum_e \beta^e + \beta^a - \beta^b - \beta^c$

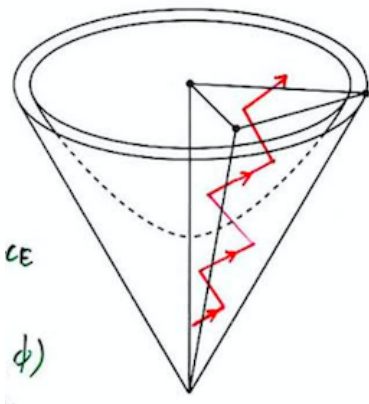
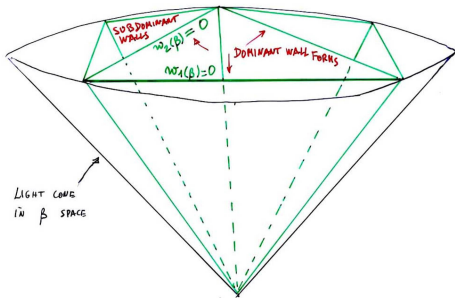
Billiard in β space: Toda-like exponential

potentials $V(\beta) = \sum_a c_A(\dots) e^{-2w_A(\beta)}$

LORENTZIAN-SIGNATURE METRIC: $G^{ab} \pi_a \pi_b \leftrightarrow G_{ab} d\beta^a d\beta^b$

$e^{-2w(\beta)} = \int \approx$ SHARP WALL $w(\beta)=0$

$$G_{ab} d\beta^a d\beta^b = \sum_{a=1}^{10} (d\beta^a)^2 - \left(\sum_{a=1}^{10} d\beta^a \right)^2$$



The appearance of E_{10} in maximal SUGRA

E_{10} , BE_{10} and Arithmetical Chaos in Superstring Cosmology

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(Received 19 December 2000)

$$S = \int d\tau \left[G_{\mu\nu} \frac{d\beta^\mu}{d\tau} \frac{d\beta^\nu}{d\tau} - V(\beta^\mu) \right]$$

$$V(\beta) \simeq \sum_A C_A e^{-2w_A(\beta)}.$$

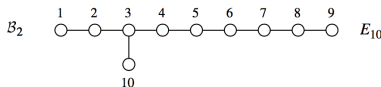
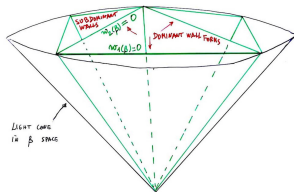
liards. The first block (with 2 \bar{SUSY} 's in $D = 10$) is $\mathcal{B}_2 = \{M, \text{IIA}, \text{IIB}\}$ and its ten walls are (in the natural variables of M theory $\beta^\mu = \beta_M^\mu$),

$$\begin{aligned} \mathcal{B}_2: w_i^{[2]}(\beta) &= -\beta^i + \beta^{i+1} \quad (i = 1, \dots, 9), \\ w_{10}^{[2]}(\beta) &= \beta^1 + \beta^2 + \beta^3. \end{aligned} \quad (4)$$

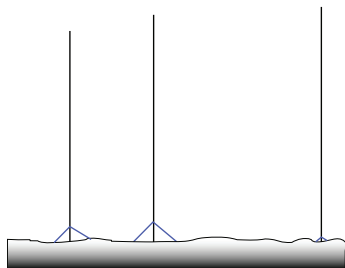
LORENTZIAN-SIGNATURE METRIC: $G^{\alpha\beta} \pi_\alpha \pi_\beta \Leftrightarrow G_{ab} d\beta^a d\beta^b$

$e^{-2w(\beta)} = \int \dots \simeq \text{SHARP WALL}$

$G_{ab} d\beta^a d\beta^b = \sum_{i=1}^9 (\beta^i)^2 - (\sum_{i=1}^9 \beta^i)^2$



From E_7 to E_{10}



Compactification down to $D = 4 \rightarrow E_7$

Compactification down to $D = 3 \rightarrow E_8$

Compactification down to $D = 2 \rightarrow E_9$

Spontaneous compactification
down to $D = 1 \rightarrow E_{10}$

Kac-Moody algebras

Generalization of the well-known “triangular” structure of $A_1 = so(3) = su(2) = sl(2)$: diagonalizable (Cartan) generator: J_z , and raising/lowering generators: $J_{\pm} = J_x \pm i J_y$ with $[J_z, J_{\pm}] = \pm J_{\pm}$; $[J_z, J_-] = -J_-$; $[J_+, J_-] = 2 J_z$

Rank r : r mutually commuting Cartan generators h_i and r simple raising (e_i) and lowering (f_i) generators:

$$[h_i, h_j] = 0; [h_i, e_j] = A_{ij} e_j; [h_i, f_j] = -A_{ij} f_j; [e_i, f_j] = \delta_{ij} h_j$$

Serre relations: $ad_{e_i}^{1-A_{ij}} e_j = 0$; $ad_{f_i}^{1-A_{ij}} f_j = 0$

$A_{ij} =$ Cartan matrix: $A_{ii} = +2$, $A_{ij} \in -\mathbb{N}$

Roots: $\alpha =$ linear form on Cartan: $h = \sum_i \beta^i h_i \rightarrow \alpha(h) = \alpha_i \beta^i$

$$E_{\alpha} \sim [e_{i_1} [e_{i_2} [e_{i_3}, \dots]]] \quad \alpha = n_1 \alpha^{(1)} + n_2 \alpha^{(2)} + \dots + n_r \alpha^{(r)}$$

$$e_i = E_{\alpha^{(i)}} \text{ simple roots; } [h, E_{\alpha}^{(s)}] = \alpha(h) E_{\alpha}^{(s)} \quad A_{ij} = \frac{2(\alpha^{(i)}, \alpha^{(j)})}{(\alpha^{(i)}, \alpha^{(i)})}$$

Dynkin Diagrams (= Cartan Matrix) of E_{10} and AE_3

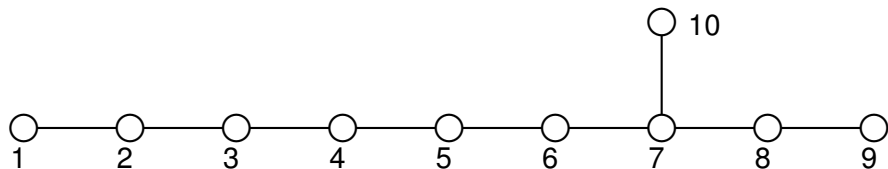
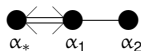


Figure: Dynkin diagram of E_{10} with numbering of nodes.

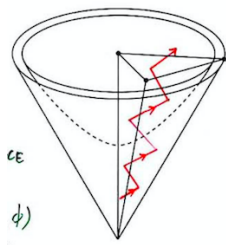
Cartan matrix of AE_3 : $(A_{ij}) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$



Dynkin diagram AE_3

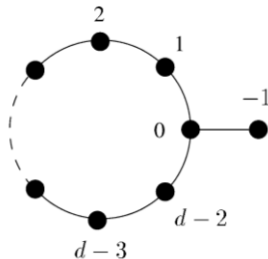
The surprising behavior of pure gravity in any D

Demaret et al 85, Damour-Henneaux-Julia-Nicolai 01

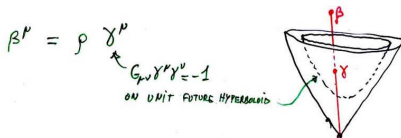
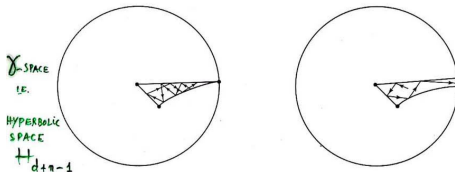
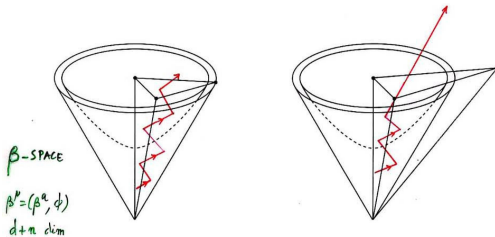


Hyperbolic Kac-Moody algebras and chaos
in Kaluza-Klein models

Thibault Damour^a, Marc Henneaux^{b,c}, Bernard Julia^d, Hermann Nicolai^e



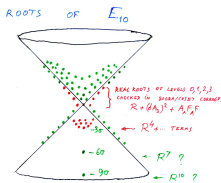
Einstein Billiards (chaotic versus non-chaotic)



Conjectured Gravity/Kac-Moody Coset Correspondence

Damour, Henneaux, Julia, Nicolai 200, [related suggestions: E_{10} , Ganor '99 '04; E_{11} : West '01]

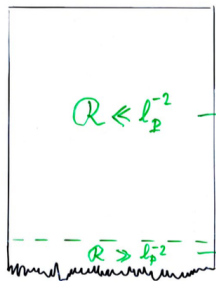
Beyond the simple roots, the potential $V(\beta) = \sum_a c_A(\dots) e^{-2w_A(\beta)}$ with $w_A(\beta) = \alpha(\beta)$ might be equivalent to including all Kac-Moody Roots $w_A(\beta) = \alpha \leftrightarrow i.e.$ all Lie-algebra generators E_α ?



Appearance of E_{10} in the “near cosmological singularity limit” suggests the existence of a supergravity/ E_{10} coset correspondence in which the ‘singularity’ is ‘resolved’ by the effective ‘disappearance’ of space, and the replacement of dynamical fields, $g_{ij}(t, \mathbf{x})$, $\mathcal{A}_{ijk}(t, \mathbf{x})$, ... by a **Lie-algebraic** variable $g(t) \in E_{10}(\mathbb{Z}) \setminus E_{10}(\mathbb{R}) / K_{10}(\mathbb{R})$

Basic Idea: two 'dual' descriptions

NEAR SPACELIKE SINGULARITY



FIELD
DESCRIPTION
SUGRA₁₁

COSET
DESCRIPTION

SUGRA₁₁ (OR M-THEORY)

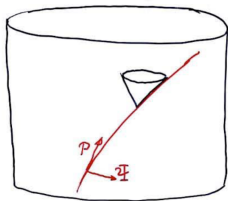
$$G_{\mu\nu}(t, \vec{x})$$

$$A_{\mu\nu\lambda}(t, \vec{x})$$

$$\psi_\mu(t, \vec{x})$$



MASSLESS SPINNING PARTICLE
ON COSET $E_{10}/K(E_{10})$



Supergravity Description

$G_{\mu\nu}(t, \mathbf{x}), \mathcal{A}_{\mu\nu\lambda}(t, \mathbf{x}), \psi_\mu(t, \mathbf{x})$
in $(T^{10}?)$ compactified space

$$\mathcal{R} \ll l_p^{-2}$$

Coset Description

$g(t) \in E_{10}(\mathbb{Z}) \backslash E_{10}(\mathbb{R}) / K_{10}(\mathbb{R})$

$$\mathcal{R} \gg l_p^{-2}$$

Gravity/coset correspondence

(super)gravity \leftrightarrow massless (spinning) particle on G/K

$g(t) \in G/K$; velocity $v \equiv \partial_t g g^{-1} \in \text{Lie}(G)$ is decomposed into $v = \mathcal{P} + \mathcal{Q}$ where $\mathcal{Q} \in \text{Lie}(K)$ and $\mathcal{P} = v^{\text{sym}} = \frac{1}{2}(v + v^T) \in \text{Lie}(G) - \text{Lie}(K)$

Coset Action for massless particle:

$$S_{\text{BOS}}^{\text{coset}} = \int \frac{dt}{n(t)} \frac{1}{4} \langle \mathcal{P}(t), \mathcal{P}(t) \rangle$$

$n(t)$: coset lapse \rightarrow constraint $\langle \mathcal{P}(t), \mathcal{P}(t) \rangle = 0$

For hyperbolic (or more generally Lorentzian) Kac-Moody algebras the coset G/K is an infinite dimensional Lorentzian space of signature $-++++\dots$

Evidence for gravity/coset correspondence

Damour, Henneaux, Nicolai 02; Damour, Kleinschmidt, Nicolai 06; de Buyl, Henneaux, Paulot 06; Kleinschmidt, Nicolai 06

Insert in $S_1^{\text{COSET}} = \int dt \left\{ \frac{1}{4n(t)} \langle P(t), P(t) \rangle - \frac{i}{2} (\Psi(t) | \mathcal{D}^{\text{vs}} \Psi(t))_{\text{vs}} + \dots \right\}$ the $GL(10)$ level expansion of the coset element

$$g(t) = \exp(h_b^a(t) K_a^b) \times \\ \times \exp \left[\frac{1}{3!} A_{abc}(t) E^{abc} + \frac{1}{6!} A_{a_1 \dots a_6}(t) E^{a_1 \dots a_6} + \frac{1}{9!} A_{a_0 | a_1 \dots a_8}(t) E^{a_0 | a_1 \dots a_8} + \dots \right].$$

Agreement (up to height 29) of EOM of $g^{ab}(t) = (e^h)_c^a (e^h)_c^b$, $A_{abc}(t)$, $A_{a_1 \dots a_6}(t)$, $A_{a_0 | a_1 \dots a_8}(t)$, and $\Psi_a^{\text{coset}}(t)$ with supergravity EOM (including lowest spatial gradients) for $G_{\mu\nu}(t, \mathbf{x})$, $\mathcal{A}_{\mu\nu\lambda}(t, \mathbf{x})$, $\psi_\mu(t, \mathbf{x})$ with dictionary:

$$g^{ab}(t) = G^{ab}(t, \mathbf{x}_0), \quad \dot{A}_{abc}(t) = \mathcal{F}_{0abc}(t, \mathbf{x}_0),$$

$$DA^{a_1 \dots a_6}(t) = -\frac{1}{4!} \varepsilon^{a_1 \dots a_6 b_1 \dots b_4} \mathcal{F}_{b_1 \dots b_4}(t, \mathbf{x}_0),$$

$$DA^{b | a_1 \dots a_8}(t) = \frac{3}{2} \varepsilon^{a_1 \dots a_8 b_1 b_2} C_{b_1 b_2}^b(t, \mathbf{x}_0)$$

$$\text{and } \Psi_a^{\text{coset}}(t) = G^{1/4} \psi_a(t, \mathbf{x}_0)$$

Moreover, \exists roots in E_{10} formally associated with the infinite towers of higher spatial gradients of $G_{\mu\nu}(t, \mathbf{x})$, $\mathcal{A}_{\mu\nu\lambda}(t, \mathbf{x})$, $\psi_\mu(t, \mathbf{x})$

Bosonic EOM of SUGRA₁₁

$D = 11$ spacetime, zero-shift slicing ($N^i = 0$) **time-independent** spatial coframe $\theta^a(x) \equiv E^a_i(x) dx^i$, $i = 1, \dots, 10$; $a = 1, \dots, 10$ choose time coordinate x^0 s.t. lapse $N = \sqrt{G}$ with $G := \det G_{ab}$

structure constants of frame: $d\theta^a = \frac{1}{2} C^a_{bc} \theta^b \wedge \theta^c$; frame derivative $\partial_a \equiv E^i_a(x) \partial_i$;
3-form \mathcal{A} ; 4-form $\mathcal{F} = d\mathcal{A}$; $2G_{ad}\Gamma^d_{bc} = C_{abc} + C_{bca} - C_{cab} + \partial_b G_{ca} + \partial_c G_{ab} - \partial_a G_{bc}$

$$ds^2 = -N^2(dx^0)^2 + G_{ab}\theta^a\theta^b$$

$$\mathcal{F} = \frac{1}{3!} \mathcal{F}_{0abc} dx^0 \wedge \theta^a \wedge \theta^b \wedge \theta^c + \frac{1}{4!} \mathcal{F}_{abcd} \theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d$$

$$\partial_0(G^{ac}\partial_0 G_{cb}) = \frac{1}{6} G\mathcal{F}^{a\beta\gamma\delta} \mathcal{F}_{b\beta\gamma\delta} - \frac{1}{72} G\mathcal{F}^{\alpha\beta\gamma\delta} \mathcal{F}_{\alpha\beta\gamma\delta} \delta_b^a - 2GR^a_b(\Gamma, C)$$

$$\begin{aligned} \partial_0(G\mathcal{F}^{0abc}) &= \frac{1}{144} \varepsilon^{abca_1 a_2 a_3 b_1 b_2 b_3 b_4} \mathcal{F}_{0a_1 a_2 a_3} \mathcal{F}_{b_1 b_2 b_3 b_4} \\ &+ \frac{3}{2} G\mathcal{F}^{de[ab} C^c]_{de} - GC^e_{de} \mathcal{F}^{dabc} - \partial_d(G\mathcal{F}^{dabc}) \end{aligned}$$

$$\partial_0 \mathcal{F}_{abcd} = 6\mathcal{F}_{0e[ab} C^e_{cd]} + 4\partial_{[a} \mathcal{F}_{0bcd]}$$

$K(E_{10})$ Structure of Gravitino Eq. of Motion

In the gauge $\psi_0^{(11)} = \Gamma_0 \Gamma^a \psi_a^{(11)}$, the equation of motion of the rescaled **gravitino** $\psi_a^{(10)} := g^{1/4} \psi_a^{(11)}$ (**neglecting cubic terms**) reads

$$\begin{aligned} \mathcal{E}_a &= \partial_t \psi_a^{(10)} + \omega_{tab}^{(11)} \psi^{(10)b} + \frac{1}{4} \omega_{tcd}^{(11)} \Gamma^{cd} \psi_a^{(10)} \\ &- \frac{1}{12} F_{tbcd}^{(11)} \Gamma^{bcd} \psi_a^{(10)} - \frac{2}{3} F_{tabc}^{(11)} \Gamma^b \psi^{(10)c} + \frac{1}{6} F_{tbcd}^{(11)} \Gamma_a{}^{bc} \psi^{(10)d} \\ &+ \frac{N}{144} F_{bcde}^{(11)} \Gamma^0 \Gamma^{bcde} \psi_a^{(10)} + \frac{N}{9} F_{abcd}^{(11)} \Gamma^0 \Gamma^{bcde} \psi_e^{(10)} - \frac{N}{72} F_{bcde}^{(11)} \Gamma^0 \Gamma_{abcdef} \psi^{(10)f} \\ &+ N(\omega_{abc}^{(11)} - \omega_{bac}^{(11)}) \Gamma^0 \Gamma^b \psi^{(10)c} + \frac{N}{2} \omega_{abc}^{(11)} \Gamma^0 \Gamma^{bcd} \psi_d^{(10)} - \frac{N}{4} \omega_{bcd}^{(11)} \Gamma^0 \Gamma^{bcd} \psi_a^{(10)} \\ &+ Ng^{1/4} \Gamma^0 \Gamma^b \left(2\partial_a \psi_b^{(11)} - \partial_b \psi_a^{(11)} - \frac{1}{2} \omega_{acb}^{(11)} \psi_a^{(11)} - \omega_{00a}^{(11)} \psi_b^{(11)} + \frac{1}{2} \omega_{00b}^{(11)} \psi_a^{(11)} \right). \end{aligned}$$

Apart from the last line, this is equivalent to the $K(E_{10})$ -covariant equation

$$0 = \overset{\text{vs}}{\mathcal{D}} \Psi(t) := \left(\partial_t - \overset{\text{vs}}{\mathcal{Q}}(t) \right) \Psi(t).$$

expressing the **parallel propagation** of the $K(E_{10})$ **vector-spinor** $\Psi(t)$ along the $E_{10}/K(E_{10})$ worldline of the coset particle, with the $K(E_{10})$ **connection** $\overset{\text{vs}}{\mathcal{Q}}(t) := \frac{1}{2}(\mathbf{v}(t) - \mathbf{v}^T(t)) \in \text{Lie}(K(E_{10}))$, with $\mathbf{v}(t) = \partial_t g g^{-1} \in \mathfrak{e}_{10} \equiv \text{Lie}(E_{10})$.

Quantum Gravity \leftrightarrow Quantum Coset Model

Quantum Bosonic Coset Model: in configuration space β^a, ν^α

$$\square_{E_{10}/K_{10}} \Psi(\beta^a, \nu^\alpha) = 0$$

$$\left[-G^{ab} \partial_{\beta^a} \partial_{\beta^b} - \sum_{\alpha > 0} e^{-2\alpha(\beta)} \partial_{\nu^\alpha}^2 + \dots \right] \Psi(\beta^a, \nu^\alpha) = 0$$

Infinite-dimensional Klein-Gordon type equation: $- + + + + \dots$

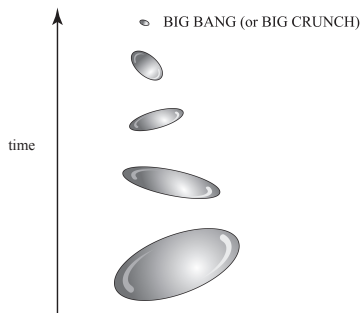
From Hull-Townsend '95 expect $E_{10}(\mathbb{Z})$ symmetry, i.e.

$\Psi(\beta, \nu) =$ automorphic function over $E_{10}(\mathbb{Z}) \backslash E_{10}(\mathbb{R}) / K_{10}(\mathbb{R})$

(Ganor '99; Brown, Ganor, Helfgott '04) Kac-Moody-Eisenstein series
(Fleig, Kleinschmidt, 12, 13)

A concrete case study (Damour, Spindel 2013, 2014, 2017)

- Quantum supersymmetric Bianchi IX, i.e. quantum dynamics of a supersymmetric **triaxially squashed three-sphere**: with squashing parameters $a = e^{-\beta^1}$, $b = e^{-\beta^2}$, $c = e^{-\beta^3}$



Susy Quantum Cosmology: Obregon et al ≥ 1990 ; D'Eath, Hawking, Obregon, 1993, D'Eath ≥ 1993 , Csordas, Graham 1995, Moniz ≥ 94 , ...

Quantum susy Bianchi IX

Technically: Reduction to one, time-like, dimension of the action of $D = 4$ simple supergravity for an $SU(2)$ -homogeneous (Bianchi IX) cosmological model \rightarrow (essentially) **Susy Quantum Mechanical model**

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2(t)dt^2 + g_{ab}(t)(\tau^a(x) + N^a(t)dt)(\tau^b(x) + N^b(t)dt),$$

τ^a : left-invariant one-forms on $SU(2) \approx S_3$: $d\tau^a = \frac{1}{2} C_{bc}^a \tau^b \wedge \tau^c$; here $C_{bc}^a = \varepsilon_{abc}$ plays the role of a nonabelian “gravitational flux”, or constant momentum of (coset) dual graviton.

Dynamical degrees of freedom

- **6 bosonic** dof: Gauss-decomposition of the metric: $g_{bc} = \sum_{\hat{a}=1}^3 e^{-2\beta^{\hat{a}}} S^{\hat{a}}_b(\varphi_1, \varphi_2, \varphi_3) S^{\hat{a}}_c(\varphi_1, \varphi_2, \varphi_3)$
 $\beta^a = (\beta^1(t), \beta^2(t), \beta^3(t))$ cologarithms of the squashing parameters a, b, c of 3-sphere $a = e^{-\beta^1}$, $b = e^{-\beta^2}$, $c = e^{-\beta^3}$ and three Euler angles: $\varphi_a = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$ parametrizing the orthogonal matrix $S^{\hat{a}}_b$

- and **12 fermionic** dof: Gravitino components in specific gauge-fixed orthonormal frame $\theta^{\hat{\alpha}}$ canonically associated with the Gauss-decomposition $\theta^{\hat{0}} = N(t)dt$, $\theta^{\hat{a}} = \sum_b e^{-\beta^a(t)} S^{\hat{a}}_b(\varphi_c(t))(\tau^b(x) + N^b(t)dt)$
- redefinitions of the gravitino field:

$$\Psi_{\hat{\alpha}}^A(t) := g^{1/4} \psi_{\hat{\alpha}}^A \quad \text{and} \quad \Phi_A^a := \sum_B \gamma_{AB}^{\hat{a}} \Psi_{\hat{a}}^B \quad (\text{no summation on } \hat{a})$$

- 3×4 gravitino components Φ_A^a , $a = 1, 2, 3$; $A = 1, 2, 3, 4$.

Supersymmetric action (first order form)

$$S = \int dt \left[\pi_a \dot{\beta}^a + p_{\varphi^a} \dot{\varphi}^a + \frac{i}{2} G_{ab} \Phi_A^a \dot{\Phi}_A^b + \bar{\Psi}_0^{I/A} S_A - \tilde{N}H - N^a H_a \right]$$

G_{ab} : Lorentzian-signature quadratic form:

$$G_{ab} d\beta^a d\beta^b \equiv \sum_a (d\beta^a)^2 - \left(\sum_a d\beta^a \right)^2$$

G_{ab} defines the kinetic terms of the gravitino, as well as those of the β^a 's:

$$\frac{1}{2} G_{ab} \dot{\beta}^a \dot{\beta}^b$$

Lagrange multipliers \longrightarrow Constraints $S_A \approx 0, H \approx 0, H_a \approx 0$

Quantization

- Bosonic dof:

$$\hat{\pi}_a = -i \frac{\partial}{\partial \beta^a} ; \quad \hat{p}_{\varphi_a} = -i \frac{\partial}{\partial \varphi_a}$$

- Fermionic dof:

$$\hat{\Phi}_A^a \hat{\Phi}_B^b + \hat{\Phi}_B^b \hat{\Phi}_A^a = G^{ab} \delta_{AB}$$

This is the Clifford algebra $\text{Spin}(8^+, 4^-)$

- The wave function of the universe $\Psi_\sigma(\beta^a, \varphi_a)$ is a 64-dimensional spinor of $\text{Spin}(8, 4)$ and the gravitino operators Φ_A^a are 64×64 “gamma matrices” acting on Ψ_σ , $\sigma = 1, \dots, 64$

- Crucially depends on the terms **cubic and quartic** in Fermions

Dirac Quantization of the Constraints

$$\widehat{S}_A \Psi = 0, \quad \widehat{H} \Psi = 0, \quad \widehat{H}_a \Psi = 0$$

Diffeomorphism constraint $\Leftrightarrow \widehat{p}_{\varphi_a} \Psi = -i \frac{\partial}{\partial \varphi_a} \Psi = 0$: “s wave” w.r.t. the Euler angles

→ Wave function $\Psi(\beta^a)$ submitted to constraints

$$\widehat{S}_A(\widehat{\pi}, \beta, \widehat{\Phi}) \Psi(\beta) = 0, \quad \widehat{H}(\widehat{\pi}, \beta, \widehat{\Phi}) \Psi(\beta) = 0$$

$\widehat{\pi}_a = -i \frac{\partial}{\partial \beta^a} \Rightarrow 4 \times 64 + 64$ PDE's for the 64 functions $\Psi_\sigma(\beta^1, \beta^2, \beta^3)$

Heavily overdetermined system of PDE's, but **consistent** by susy:

$$\widehat{S}_A \widehat{S}_B + \widehat{S}_B \widehat{S}_A = 4i \sum_C \widehat{L}_{AB}^C(\beta) \widehat{S}_C + \frac{1}{2} \widehat{H} \delta_{AB}$$

Explicit form of the SUSY constraints ($\gamma^5 \equiv \gamma^{\hat{0}\hat{1}\hat{2}\hat{3}}$, $\beta_{12} \equiv \beta^1 - \beta^2$, $\hat{\Phi}^{12} \equiv \hat{\Phi}^1 - \hat{\Phi}^2$)

$$\begin{aligned} \hat{\mathcal{S}}_A &= -\frac{1}{2} \sum_a \hat{\pi}_a \Phi_A^a + \frac{1}{2} \sum_a e^{-2\beta^a} (\gamma^5 \Phi^a)_A \\ &\quad - \frac{1}{8} \coth \beta_{12} (\hat{\mathcal{S}}_{12} (\gamma^{12} \hat{\Phi}^{12})_A + (\gamma^{12} \hat{\Phi}^{12})_A \hat{\mathcal{S}}_{12}) \\ &\quad + \text{cyclic}_{(123)} + \frac{1}{2} (\hat{\mathcal{S}}_A^{\text{cubic}} + \hat{\mathcal{S}}_A^{\text{cubic} \dagger}) \end{aligned}$$

$$\begin{aligned} \hat{\mathcal{S}}_{12}(\hat{\Phi}) &= \frac{1}{2} [(\tilde{\hat{\Phi}}^3 \gamma^{\hat{0}\hat{1}\hat{2}} (\hat{\Phi}^1 + \hat{\Phi}^2)) + (\tilde{\hat{\Phi}}^1 \gamma^{\hat{0}\hat{1}\hat{2}} \hat{\Phi}^1) \\ &\quad + (\tilde{\hat{\Phi}}^2 \gamma^{\hat{0}\hat{1}\hat{2}} \hat{\Phi}^2) - (\tilde{\hat{\Phi}}^1 \gamma^{\hat{0}\hat{1}\hat{2}} \hat{\Phi}^2)], \end{aligned}$$

$$\begin{aligned} \hat{\mathcal{S}}_A^{\text{cubic}} &= \frac{1}{4} \sum_a (\tilde{\Psi}_0, \gamma^{\hat{0}} \hat{\Psi}_{\hat{a}}) \gamma^{\hat{0}} \hat{\Psi}_{\hat{a}}^A - \frac{1}{8} \sum_{a,b} (\tilde{\Psi}_{\hat{a}} \gamma^{\hat{0}} \hat{\Psi}_{\hat{b}}) \gamma^{\hat{a}} \hat{\Psi}_{\hat{b}}^A \\ &\quad + \frac{1}{8} \sum_{a,b} (\tilde{\Psi}_0, \gamma^{\hat{a}} \hat{\Psi}_{\hat{b}}) (\gamma^{\hat{a}} \hat{\Psi}_{\hat{b}}^A + \gamma^{\hat{b}} \hat{\Psi}_{\hat{a}}^A), \end{aligned}$$

$AE_3 = A_1^{++}$ linked to $a = e^{-\beta^1}$, $b = e^{-\beta^2}$, $c = e^{-\beta^3}$

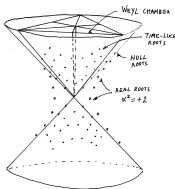
Cartan matrix of AE_3 : $(A_{ij}) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

3-dimensional Lorentzian-signature space: metric in Cartan sub-algebra

$$G_{ab} d\beta^a d\beta^b = \sum_a (d\beta^a)^2 - \left(\sum_a d\beta^a \right)^2$$

Simple roots linked to symmetry and gravitational walls

$$\alpha_{12} = \beta^2 - \beta^1 ; \alpha_{23} = \beta^3 - \beta^2 ; \alpha_{11} = 2\beta^1$$



Kac-Moody Structures Hidden in the Quantum Hamiltonian

$$2\hat{H} = G^{ab}(\hat{\pi}_a + iA_a)(\hat{\pi}_b + iA_b) + \hat{\mu}^2 + W_g^{\text{bos}}(\beta) + \widehat{W}_g^{\text{spin}}(\beta) + \widehat{W}_{\text{sym}}^{\text{spin}}(\beta).$$

$G_{ab} \leftrightarrow$ metric in Cartan subalgebra of AE_3

$$W_g^{\text{bos}}(\beta) = \frac{1}{2} e^{-2\alpha_{11}^g(\beta)} - e^{-2\alpha_{23}^g(\beta)} + \text{cyclic}_{123}$$

$$\widehat{W}_g^{\text{spin}}(\beta, \widehat{\Phi}) = e^{-\alpha_{11}^g(\beta)} \widehat{J}_{11}(\widehat{\Phi}) + e^{-\alpha_{22}^g(\beta)} \widehat{J}_{22}(\widehat{\Phi}) + e^{-\alpha_{33}^g(\beta)} \widehat{J}_{33}(\widehat{\Phi}).$$

Linear forms $\alpha_{ab}^g(\beta) = \beta^a + \beta^b \Leftrightarrow$ six level-1 roots of AE_3

$$\widehat{W}_{\text{sym}}^{\text{spin}}(\beta) = \frac{1}{2} \frac{(\widehat{S}_{12}(\widehat{\Phi}))^2 - 1}{\sinh^2 \alpha_{12}^{\text{sym}}(\beta)} + \text{cyclic}_{123},$$

Linear forms $\alpha_{12}^{\text{sym}}(\beta) = \beta^1 - \beta^2$, $\alpha_{23}^{\text{sym}}(\beta) = \beta^2 - \beta^3$, $\alpha_{31}^{\text{sym}}(\beta) = \beta^3 - \beta^1$
 \Leftrightarrow three level-0 roots of AE_3

Spin dependent (Clifford) Operators coupled to AE_3 roots

$$\begin{aligned}\widehat{S}_{12}(\widehat{\Phi}) &= \frac{1}{2} [(\widehat{\Phi}^3 \gamma^{\widehat{0}\widehat{1}\widehat{2}}(\widehat{\Phi}^1 + \widehat{\Phi}^2)) + (\widehat{\Phi}^1 \gamma^{\widehat{0}\widehat{1}\widehat{2}} \widehat{\Phi}^1) \\ &+ (\widehat{\Phi}^2 \gamma^{\widehat{0}\widehat{1}\widehat{2}} \widehat{\Phi}^2) - (\widehat{\Phi}^1 \gamma^{\widehat{0}\widehat{1}\widehat{2}} \widehat{\Phi}^2)],\end{aligned}$$

$$\widehat{J}_{11}(\widehat{\Phi}) = \frac{1}{2} [\widehat{\Phi}^1 \gamma^{\widehat{1}\widehat{2}\widehat{3}} (4\widehat{\Phi}^1 + \widehat{\Phi}^2 + \widehat{\Phi}^3) + \widehat{\Phi}^2 \gamma^{\widehat{1}\widehat{2}\widehat{3}} \widehat{\Phi}^3].$$

- \widehat{S}_{12} , \widehat{S}_{23} , \widehat{S}_{31} , \widehat{J}_{11} , \widehat{J}_{22} , \widehat{J}_{33} generate (via commutators) a 64-dimensional representation of the (infinite-dimensional) “maximally compact” sub-algebra $K(AE_3) \subset AE_3$. [The fixed set of the (linear) Chevalley involution, $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $\omega(h_i) = -h_i$, which is generated by $x_i = e_i - f_i$.]

The “squared-mass” Quartic Operator $\widehat{\mu}^2$ in \widehat{H}

In the middle of the Weyl chamber (far from all the hyperplanes $\alpha_i(\beta) = 0$):

$$2\widehat{H} \simeq \widehat{\pi}^2 + \widehat{\mu}^2$$

where $\widehat{\mu}^2 \sim \sum \widehat{\Phi}^4$ gathers many complicated **quartic-in-fermions** terms (including $\sum \widehat{S}_{ab}^2$ and the infamous ψ^4 terms of supergravity).

Remarkable Kac-Moody-related facts:

- $\widehat{\mu}^2$ **commutes** with the $K(AE_3)$ generators $\widehat{S}_{ab}, \widehat{J}_{ab}$
- $\widehat{\mu}^2$ is \sim the square of a very simple operator \in Center

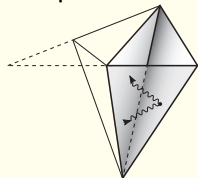
$$\widehat{\mu}^2 = \frac{1}{2} - \frac{7}{8} \widehat{C}_F^2$$

where $\widehat{C}_F := \frac{1}{2} G_{ab} \widehat{\Phi}^a \gamma^{\hat{1}\hat{2}\hat{3}} \widehat{\Phi}^b$.

Solutions of SUSY constraints

Overdetermined system of **four** 64-component Dirac-like equations

$$\widehat{S}_A \Psi = \left(\frac{i}{2} \Phi_A^a \frac{\partial}{\partial \beta^a} + U(\beta) \Phi \Phi \Phi \right) \Psi = 0$$



Space of solutions is a mixture of “**discrete-spectrum states**” and “**continuous-spectrum states**”, depending on fermion number $N_F = C_F - 3$. \exists solutions for both even and odd N_F .

\exists continuous-spectrum states (parametrized by initial data comprising arbitrary *functions*) at $C_F = -1, 0, +1$.

Completion of inconclusive studies started long ago: D’Eath 93, D’Eath-Hawking-Obregon 93, Csordas-Graham 95, Obregon 98, ...

Solution space of quantum susy Bianchi IX: $N_F = 0$

Level $N_F = 0$: \exists unique “ground state” $|f\rangle = C f_0(\beta) |0\rangle_-$ with

$$f_0(\beta) = abc \left[(b^2 - a^2)(c^2 - b^2)(c^2 - a^2) \right]^{3/8} e^{-\frac{1}{2}(a^2+b^2+c^2)} |0\rangle_-$$

This “ground state” is localized in the middle of β space (or of a Weyl chamber) and decays in all directions in β space: small volume, large volume, large anisotropies. It describes a quantum universe which oscillates in shape and size, but stays of Planckian size

\exists similar “discrete-spectrum” states at $N_F = 1, 2, 4, 5, 6$; however, it is only at levels $N_F = 0$ and 1 that these states decay in all directions and are square integrable at the symmetry walls.

Continuous-spectrum solutions at $N_F = 2, 3, 4$: Quantum Supersymmetric Billiard

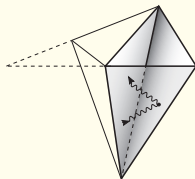
At $N_F = 2$, \exists solutions of the type $k_{(ab)}(\beta) \tilde{b}_+^a \tilde{b}_-^b |0\rangle_-$ with a symmetric $k_{ab}(\beta) = f_{(ab)}^{+-}(\beta)$, with 6 components, that satisfies Maxwell-type equations in β space similar to $\delta k \sim 0$, $dk \sim 0$.

The spinorial wave function of the universe $\Psi(\beta^a)$ propagates within the (various) Weyl chamber(s) and “reflects” on the walls (= simple roots of AE_3). In the small-wavelength limit, the “reflection operators” define a **spinorial extension of the Weyl group of AE_3** (Damour Hillmann 09) defined within some subspaces of $\text{Spin}(8, 4)$

$$\widehat{\mathcal{R}}_{\alpha_i} = \exp\left(-i \frac{\pi}{2} \widehat{\varepsilon}_{\alpha_i} \widehat{J}_{\alpha_i}\right)$$

with

$$\widehat{J}_{\alpha_i} = \{\widehat{S}_{23}, \widehat{S}_{31}, \widehat{J}_{11}\} \text{ and } \widehat{\varepsilon}_{\alpha_i}^2 = \text{Id}$$



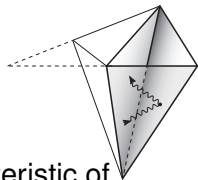
Hidden Kac-Moody structure of the spinor reflection operators

Dynamically computed reflection operators:

$$\mathcal{R}_{\alpha_{23}}^{6, N_F=2, WKB} = e^{-\frac{i\pi}{2}} e^{\pm \frac{i\pi}{2}} |\widehat{S}_{23}|_{6, N_F=2},$$

$$\mathcal{R}_{\alpha_{12}}^{6, N_F=2, WKB} = e^{-\frac{i\pi}{2}} e^{\pm \frac{i\pi}{2}} |\widehat{S}_{12}|_{6, N_F=2},$$

$$\mathcal{R}_{\alpha_{11}}^{6, N_F=2} = e^{-i\frac{\pi}{2}} e^{-i\frac{\pi}{2}} \widehat{J}_{11}^{6, N_F=2},$$



The \mathcal{R}_{α_i} 's satisfy generalized Coxeter relations characteristic of **spin-extended Weyl groups** (Ghatei, Horn, Köhl, Weiss, 2016)

$$r_i^8 = 1;$$

$r_i r_j r_i \cdots = r_j r_i r_j \cdots$ "braid" relations with m_{ij} factors on each side.

$$r_j^{-1} r_i^2 r_j = r_i^2 r_j^{2n_{ij}},$$

with some (precisely defined) integers m_{ij} and n_{ij} .

Fermions and their dominance near the singularity

Crucial issue of boundary condition near a big bang or big crunch or black hole singularity: DeWitt 67, Vilenkin 82 ..., Hartle-Hawking 83 ..., ..., Horowitz-Maldacena 03

Finding in Bianchi IX SUGRA: the WDW squared-mass term $\hat{\mu}^2$ is *negative* (i.e. tachyonic) in most of the Hilbert space (44 among 64).

$$\mu^2 = \left(-\frac{59}{8} \Big|_0^1, -3 \Big|_1^6, -\frac{3}{8} \Big|_2^{15}, +\frac{1}{2} \Big|_3^{20}, -\frac{3}{8} \Big|_4^{15}, -3 \Big|_5^6, -\frac{59}{8} \Big|_6^1 \right) \quad (1)$$

This is a quantum effect quartic in fermions:

$$\rho_4 \sim \psi^4 \sim \mu^2 (\mathcal{V}_3)^{-2} = \mu^2 (abc)^{-2} = \mu^2 \bar{a}^{-6}$$

which dominates the other contributions near a small volume singularity

Bouncing Universes and Quantum Boundary Conditions at a Spacelike Singularity ?

A "stiff", $\rho_4 = p_4$, negative $\rho_4 < 0$ contribution classically leads to an **avoidance of a singularity**, i.e. a **bounce** of the universe. This bounce is exemplified by the $N_F = 0$ "ground state"

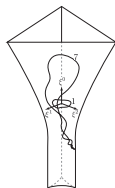
$$\Psi_0 = (abc) \left[(b^2 - a^2)(c^2 - b^2)(c^2 - a^2) \right]^{3/8} e^{-\frac{1}{2}(a^2 + b^2 + c^2)} \Big|_0$$

Quantum mechanically, the general solution of the WDW equation (in "hyperbolic polar coordinates" $\beta^a = \rho\gamma^a$)

$$\left(\frac{1}{\rho} \partial_\rho^2 \rho - \frac{1}{\rho^2} \Delta_\gamma + \hat{\mu}^2 \right) \Psi'(\rho, \gamma^a) = 0$$

behaves, after a **quantum-billiard** mode-expansion $\Psi'(\rho, \gamma^a) = \sum_n R_n(\rho) Y_n(\gamma^a)$, as

$$\rho R_n(\rho) \equiv u_n(\rho) \approx a_n e^{-|\mu|\rho} + b_n e^{+|\mu|\rho}, \text{ as } \rho \rightarrow +\infty$$



This suggests to impose the boundary condition $\Psi' \sim e^{-|\mu|\rho} \rightarrow 0$ at the singularity, which is, for a black hole crunch, a type of "final-state" boundary condition (à la Horowitz-Maldacena), which would represent a **quantum avoidance of the singularity** ?

Conclusions

- The dynamics of gravity near a cosmological singularity is chaotic and equivalent to billiard motion in the Weyl chamber of an hyperbolic Kac-Moody algebra.
- The evidence for a hidden Kac-Moody structure goes much beyond the billiard limit (both in bosonic and fermionic EOM and in classical/quantum effects). It suggests a gravity/coset correspondence: gravity dynamics \leftrightarrow massless particle on infinite-dimensional (Lorentzian-signature) Kac-Moody coset G/K .
- The case study of the quantum dynamics of a triaxially squashed 3-sphere (Bianchi IX model) in (simple, $D = 4$) supergravity confirms the hidden presence of hyperbolic Kac-Moody structures (AE_3 and $K(AE_3)$) in supergravity, especially in the fermionic sector.
- Quartic terms in the gravitino might lead to a quantum avoidance of the singularity.
- The largest hyperbolic Kac-Moody algebra E_{10} seems to play an important, though still mysterious, role in $SUGRA_{11}$.