The antifield-BRST approach to (gauge) field theories: an overview

Marc Henneaux

Exceptional Dimensions
A Conference in Honour of Bernard Julia
(16-17 December 2019, Institut Henri Poincaré)
Introduction

Bernard has always emphasized the crucial importance of cohomological ideas in physics. My talk will be an illustration of the power of cohomological methods in the context of gauge field theories. It will be devoted to BRST theory. I will focus in particular on the cohomological significance of the antifields, which is crucial for computing explicitly the BRST cohomology. These were introduced by Zinn-Justin, and Batalin and Vilkovisky.
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- BRST differential in Yang-Mills theory
- Antifields and Koszul-Tate resolution
- Homological perturbation theory (and $L^{\infty}$ algebras)
- Local functionals and characteristic complex

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Yang-Mills theory

The BRST differential in Yang-Mills theory reads (in the "minimal sector")

\[ sA_a^{\mu} = D^{\mu}C_a, \]
\[ sC_a = -\frac{1}{2} f^{a}_{bc} C_b C_c, \]
\[ sA^*_{\mu} a = D^\nu F^\nu_{\mu} a + f^b_{ac} A^*_{\mu} b C_c, \]
\[ sC^* a = D^\mu A^*_{\mu} a + f^b_{ac} C^* b C_c. \]

It is generated in the antibracket by the solution \( S \) of the "master equation",

\[ sF = (S, F) \]
\[ S = -\frac{1}{4} \int d^n x F_{\mu \nu} a F^a_{\mu \nu} + \int d^n x A^*_{\mu} a sA^a_{\mu} + \int d^n x C^* a sC^a, \]

and nilpotency of \( s \) is equivalent to the "master equation",

\[ s^2 = 0 \iff (S, S) = 0. \]
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Antifields

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Antifields

$A^*\mu$ and $C^*a$ are the "antifields". Antifields were originally introduced by Zinn-Justin in his seminal work on the renormalization of gauge theories, as sources coupled to the BRST variations of the fields. This was motivated by the desire to control how the nonlinear BRST symmetry passes through the renormalization process. A different interpretation of the antifields can be developed. This interpretation has cohomological origins and views the antifields as the generators of a differential complex that implements the gauge invariant equations of motion in cohomology. This different point of view turns out to be crucial for computing the BRST cohomology.
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The phase space $\Pi$ of a gauge theory can be covariantly described as the space of solutions to the equations of motion modulo the gauge transformations. The equations of motion define a "surface" in the space $J$ of all histories, which is called the "stationary surface" and denoted by $\Sigma$. $C^\infty(\Sigma)$ is the space of smooth functions on that surface. Formally, $\Pi$ is the quotient space $\Pi = \Sigma / O$ of the stationary surface $\Sigma$ by the gauge orbits $O$ generated by the gauge transformations. [For local objects, jet space formalism can be used to put these considerations on a firmer footing.]
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Covariant phase space

The observables are the functions on Π. This description of the observables involves two steps:

1) Restriction to the stationary surface;
2) Implementation of the gauge invariance condition on Σ.

The BRST differential provides a cohomological formulation of C∞(Π) at ghost number zero, H0(s) = {Observables}.

To each of the steps (1), (2) corresponds a separate differential. Both differentials appear in s.
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To each of the steps (1), (2) corresponds a separate differential. Both differentials appear in $s$. 

Antifield number

To exhibit this property, it is useful to introduce the antifield number,

\[ \text{puregh} \]

\[ \text{antifd} \]

\[ A^{\mu} a_0 \]

\[ C^a_1 \]

\[ A^{* \mu} a_0 \]

\[ C^{* a}_0 \]

Pure ghost number, antifield number and gh \( \equiv \text{puregh} - \text{antifd} \) \( = \) \( \text{total ghost number} \), for the different field types

One has

\[ s = \delta + \gamma \]

\[ \text{antifd}(\delta) = -1 \]

\[ \text{antifd}(\gamma) = 0 \]

Explicitly,

\[ \delta A^{\mu} a_0 = 0 \]

\[ \delta C^a_1 = 0 \]

\[ \delta A^{* \mu} a_0 = D^{\nu} F^{\nu \mu} a_0 \]

\[ \delta C^{* a}_0 = D^{\mu} A^{* \mu} a_0 \]

\[ \gamma A^{\mu} a_0 = D^{\mu} C^a_1 \]

\[ \gamma C^a_1 = -\frac{1}{2} f^{a} bc C^b C^c \]

\[ \gamma A^{* \mu} a_0 = f^{b} ac A^{* \mu} b C^c \]

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Nilpotency of \( s \) implies

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<tbody>
<tr>
<td>$A_\mu^a$</td>
<td>0</td>
<td>0</td>
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<tr>
<td>$C^a$</td>
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<td>$A^*_{\mu}^a$</td>
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Pure ghost number, antifield number and $gh \equiv$ puregh $-$ antifd (“total ghost number”), for the different field types

One has $s = \delta + \gamma$, with antifd(\(\delta\)) = $-1$ and antifd(\(\gamma\)) = 0
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<tr>
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Pure ghost number, antifield number and $gh \equiv puregh - antifd$ (“total ghost number”), for the different field types

One has $s = \delta + \gamma$, with $antifd(\delta) = -1$ and $antifd(\gamma) = 0$

Explicitly, $\delta A^a_\mu = 0$, $\delta C^a = 0$, $\delta A^*_\mu = D_\nu F^\nu_\mu$, $\delta C^*_a = D_\mu A^*_\mu$
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Explicitly, $\delta A_{\mu}^a = 0, \delta C^a = 0, \delta A_{a}^{*\mu} = D_{\nu} F_{\nu \mu}^a, \delta C_{a}^{*} = D_{\mu} A_{a}^{*\mu}$

and $\gamma A_{\mu}^a = D_{\mu} C^a, \gamma C^a = -\frac{1}{2} f_{bc}^a C^b C^c, \gamma A_{a}^{*\mu} = f_{bc}^b A_{b}^{*\mu} C^c, \gamma C_{a}^{*} = f_{bc}^b C_{b}^{*} C^c.$
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Explicitly, $\delta A^a_\mu = 0$, $\delta C^a = 0$, $\delta A^{*\mu}_a = D_\nu F^\nu_\mu$, $\delta C^{*}_a = D_\mu A^{*\mu}_a$

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Nilpotency of $s$ implies $\delta^2 = 0$, $\delta \gamma + \gamma \delta = 0$, $\gamma^2 = 0$. 
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Koszul-Tate differential

The differential $\delta$ is called the "Koszul-Tate differential" because it is associated with the Koszul-Tate resolution of the algebra of functions on the stationary surface (first step), in the sense that $H_m \equiv (\text{Ker} \delta \cap \text{Im} \delta)^m = 0$ for $m > 0$ and $H_0(\delta) = C^\infty(\Sigma)$.

The differential $\gamma$ is called the "exterior derivative along the gauge or orbits" and implements the second (gauge invariance) condition, so that $H_0(\gamma, C^\infty(\Sigma)) = \{\text{Observables}\}$.

This second aspect is well appreciated (Chevalley-Eilenberg differential and "Lie algebra cohomology" in the relevant representation space). Furthermore, it is also clear that $H_0(s) \cong H_0(H_0(\gamma), H_0(\delta))$ (standard spectral sequence argument).

We shall for this reason only focus here on the Koszul-Tate differential $\delta$. 
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This second aspect is well appreciated (Chevalley-Eilenberg differential and “Lie algebra cohomology" in the relevant representation space).
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We shall for this reason only focus here on the Koszul-Tate differential $\delta$. 

Koszul-Tate differential
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The algebra $\mathcal{C}^\infty(\Sigma)$ of smooth functions on the stationary surface can be viewed as the quotient of the algebra $\mathcal{C}^\infty(J)$ of smooth functions of the histories by the ideal $N$ of functions that vanish on $\Sigma$.

The ideal $N$ is generated by the left-hand sides $D_\nu F_{\mu\nu}^a$ of the equations of motion and their successive derivatives $\partial_\rho D_\nu F_{\mu\nu}^a$, $\partial_\sigma \partial_\rho D_\nu F_{\mu\nu}^a$, in the sense that $f \in N \iff f = k^a A^*_{\mu} D_\nu F_{\mu\nu}^a + k^a \rho_{\mu} \partial_\rho A^*_{\mu} D_\nu F_{\mu\nu}^a + k^a \rho_\sigma \partial_\sigma \partial_\rho A^*_{\mu} D_\nu F_{\mu\nu}^a + \cdots$ for some smooth coefficients $k$'s.

But this is exactly equivalent to $f = \delta h$ with $h = k^a A^*_{\mu} + k^a \rho_{\mu} \partial_\rho A^*_{\mu} + k^a \rho_\sigma \partial_\sigma \partial_\rho A^*_{\mu} + \cdots$.
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Koszul-Tate differential

Thus $\text{Im} \delta^0 = N$ and therefore $H_0(\delta) = C_\infty(\Sigma)$.

Is there (co)homology at other values of the antifield number? At antifield number 1, one finds that $D \mu A^* \mu a$ is a cycle, $\delta D \mu A^* \mu a = 0$ because of the Noether identity $D \mu D \nu F_{\mu \nu} a = 0$.

Without the antifields $C^* a$ conjugate to the ghosts, these cycles would be non trivial because they do not vanish on $\Sigma$. The antifields $C^* a$ kill these (otherwise non-trivial) cycles, so that $H_1(\delta) = 0$. Indeed, $D \mu A^* \mu a = \delta C^* a$. 

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Indeed,

$$D_\mu A^*_a \mu = \delta C^*_a.$$
Koszul-Tate differential

One can show that similarly, $H_m(\delta) = 0$, ($m \geq 1$). (If the gauge transformations were reducible, one would need "ghosts of ghosts" and on the Koszul-Tate side, "antifields for antifields"). Thus, the Koszul-Tate complex provides a resolution of the algebra $\mathcal{C}_\infty(\Sigma)$ of smooth functions on the stationary surface. (If one includes the ghosts, one gets $\mathcal{C}_\infty(\Sigma) \otimes \Lambda(C^a, \partial^\mu C^a, \cdots)$.)
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Beyond Yang-Mills

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The recognition of the antifields as related to a resolution of the stationary surface is key to the formulation of BRST theory beyond Yang-Mills.

(1) When the gauge transformations are reducible, one needs ghosts of ghosts and their conjugate antifields to maintain the resolution property.

(2) When the gauge transformations are “open” (on-shell closure only), the construction is more elaborate because $\gamma_2 \neq 0$, but $\gamma_2 \approx 0$ (only on-shell). This requires additional terms in $s$:

$$s = \delta + \gamma + s_1 + s_2 + \cdots$$

To guarantee $s_2 = 0$.

This is the Batalin-Vilkovisky construction, which works because the Koszul-Tate complex is a resolution.
Beyond Yang-Mills

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To give an idea:
\[
\delta + \gamma + \cdots \equiv (\delta + \gamma + \cdots)^2 = \delta^2 + (\delta \gamma + \gamma \delta) + \gamma^2 + \cdots \equiv 0 + 0 + 0 + \cdots,
\]
but one has\[\gamma^2 = -\delta s_1 - s_1 \delta\]for some\[s_1\]and therefore,
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\delta + \gamma + s_1 + \cdots \equiv (\delta + \gamma + s_1 + \cdots)^2 = \delta^2 + (\delta \gamma + \gamma \delta) + (\gamma^2 + \delta s_1 + s_1 \delta) + \cdots \equiv 0 + 0 + 0 + \cdots.
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$p$-form gauge fields, ghosts of ghosts

$B_{\mu\nu}, C_\mu, \gamma$

$B_{\mu\nu} = \partial_{[\mu} C_{\nu]}$

$\delta C^*_{\mu} = \partial_{\nu} B^*_{\mu\nu}$

$\delta\partial_{\mu} C^*_{\mu} = 0$

Need to introduce antifield $C^*$ at antifield number $-3$ such that $\delta C^*_{\mu} = \partial_{\mu} C^*_{\mu}$.

On the ghost side, needs conjugate "ghost of ghosts" $C$ with ghost number $+2$ and such that $\gamma C_{\mu} = \partial_{\mu} C$.

Procedure works and corresponding term in the solution of the master equation is $\sim \int d^n x C^*_{\mu} \partial_{\mu} C$. 
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Procedure works and corresponding term in the solution of the master equation is \( \sim \int d^n x C^*{}^\mu \partial_\mu C. \)
Master equation

In terms of the solution $S$ of the master equation $(S, S) = 0$, $S = S_0 + S_1 + S_2 + \cdots$ with $(S_0, S_0) = 0$, $(S_0, S_1) = 0$, $2(S_0, S_2) + (S_1, S_1) = 0$, $\cdots$.

Acyclicity of $\delta$ guarantees the existence of $S_2$ and of the successive terms. For instance, $(S_0, S_1) = 0$ implies $(S_0, (S_1, S_1)) = 0$ from which one infers the existence of $S_2$ using acyclicity. Etc.
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Etc
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In that case, however, there is in general no matching between fields and antifields and no natural antibracket.

The BRST differential $s = \delta + \gamma + \cdots$ can be constructed as before but is not generated in the antibracket through the solution $S$ of the master equation.
Local Functionals

Also quite relevant to physics is the BRST cohomology in the space of local functionals. A local functional is the integral of a local $n$-form. A local $p$-form is a $p$-form with coefficients that are local functions. So a local functional is $F = \int \omega$, where $\omega = f d^{n}x$, and $f$ is a local function.
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Local Functionals

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A local functional is the integral of a local $n$-form

A local $p$-form is a $p$-form with coefficients that are local functions.
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So a local functional is

$$F = \int \omega, \quad \omega = fd^n x,$$

where $f$ is a local function.
BRST cohomology in the space of local functionals

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The cocycles and coboundary conditions for local functionals read

\[ \omega + d \alpha = 0 \]

and

\[ \omega = s \psi + db \]
in terms of the integrands since

\[ \int d \alpha = 0. \]

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This defines the mod-$d$ cohomology $H^m(s|d)$. 
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it turns out not to be true that $H_m(s|d) = 0$ for $m > 0$.

For instance an abelian ghost $C^*$ fulfills $\delta C^* = \partial \mu A^*\mu$ and defines a non-trivial element of $H_2(\delta|d)$. 
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Characteristic cohomology

The characteristic cohomology in form-degree \(k\) is defined to be the space of \(k\)-forms that are closed on-shell modulo the \(k\)-form that are exact on-shell, \(da \approx 0, a \sim a'\) iff \(a' - a \approx db\).

The conserved currents correspond to the characteristic cohomology in form-degree \(n-1\).

Using the Koszul-Tate differential, one easily sees that the characteristic cohomology is just

\[
H^0_k (d|\delta) \approx H^0_{k-1} (d|\delta), \quad i, j > 1, (i, j) \neq (1, 1);
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Using the Koszul-Tate differential, one easily sees that the characteristic cohomology is just $H^k_0(d|\delta)$.

Reading the cocycle condition $\delta \omega + da = 0$ in both directions, one easily proves the isomorphisms

$$H^i_j(\delta|d) \simeq H^{i-1}_{j-1}(d|\delta), \quad i,j > 1, \ (i,j) \neq (1,1); \quad H^1_1(\delta|d) \simeq \frac{H^0_0(d|\delta)}{\mathbb{R}}.$$
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Once this is done, one can compute explicitly the BRST cohomology \( H(s|d) \) (BRST cohomology in the space of local functionals).
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Once this is done, one can compute explicitly the BRST cohomology $H(s|d)$ (BRST cohomology in the space of local functionals).

The understanding that $s$ involves $\delta$ – and the corresponding spectral sequence – is crucial for this purpose.
Conclusions and comments

Antifields were originally introduced by Zinn-Justin as sources coupled to the BRST variations of the fields. A different interpretation of the antifields can be developed. The antifields can indeed also be viewed as the generators of the Koszul-Tate "resolution" that implements the equations of motion in cohomology. This point of view turns out to be crucial for computing the BRST cohomology.
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THANK YOU!
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