Some Remarks on Supergravity by a Geometer

Conference to honour Bernard JULIA

Paris, 16 December 2019

A Possible Subtitle

- The purpose of the lecture is to exhibit ways in which the Theory of Supergravity stimulated Mathematics: *notably through the exploration of specific geometric objects of importance in Riemannian Geometry*.
- In this lecture I will try and connect with the seminal work of Eugène CREMMER, Bernard JULIA and Joël SCHERK, but also with some other works in Theoretical Physics

Cremmer-Julia-Scherk 11-Dimensional Supergravity

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SUPERGRAVITY THEORY IN 11 DIMENSIONS

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We present the action and transformation laws of supergravity in 11 dimensions which is expected to be closely related to the O(8) theory in 4 dimensions after dimensional reduction.

Extended O(N) (N= 1, ... 8) supergravity theories [1–5] are notoriously difficult to construct beyond N = 3. The difficulty lies partly in the large number of fields involved (for N = 8, which is the largest theory that can be constructed in this frame-work, one has 1 graviton, 8 spin 3/2 gravitinos, 28 vectors, 56 spinors, 35 scalar and 35 pseudoscalar particles) but mostly in the fact that the spin 0 fields appear in a non-polynomial way, thus forbidding a step-by-step construction of the action and transformation laws. So far, only the N = 4 theory has been constructed in a closed form [3], the simplest form of it exhibiting a manifest SU(4)

N = 4 theory, which has a vanishing β-function at the first two non-trivial orders [8] was found, and also a systematic search of all supersymmetric Yang–Mills in less than 10 dimensions was conducted [9].

As shown by Nahm [10], D = 10 is the highest number of dimensions in which supersymmetry representations with $J \leqslant 1$ can exist, while supergravity theories ($J \leqslant 2$) can exist up to D = 11. The interest in constructing the 11 dimensional theory lies in the fact that its reduction to four dimensions is automatically guaranteed to yield an O(7) invariant supergravity theory which has exactly the same field content as the O(8)

Summary

- Supergravity
- The view of a mathematician on spinors
- Killing Spinors
- Holonomy
- Perspectives

1. Supergravity

General Relativity and Quantum Physics

Developing a theory that would encompass General Relativity and Quantum Mechanics has been considered a key challenge of Physics almost from the onset of the two theories:

- Paul-Adrien-Maurice DIRAC introduced the concept of *spinor* in Physics precisely to give a relativistically invariant version of the Schrödinger equation;
- A first attempt to enlarge space-time to give a natural setting to couple gravitation and other interactions was made by Theodor KALUZA;
- It was brought to another level by the development of non-Abelian gauge theories that could deal with interactions other than gravity;
- The purpose of developing a theory of Supergravity was precisely to bridge this gap.

Supergravity as Physical Theory

The celebration of 40 years of Supergravity a few years ago of course gave rise to a number of articles looking back at its history:

- The theory has generated tens of thousands of articles;
- Its fate has of course been linked to the fate of supersymmetry in its several disguises: String Theory, M-Theory, ...;
- The informations on the Higgs Boson given to this point by LHC experiments seem to indicate that the most obvious way supersymmetric theories could complement the Standard Model does not work and more thought has to be given to the issue;
- The Breakthrough Prize given to Peter NIEUWENHUIZEN, Sergio FERRARA and Daniel FREEDMAN for their seminal paper on Supergravity Theory last year led to further contestations but I will come back to this in my conclusion.

The Mathematical Content of Supergravity

For me as a Geometer, and I am conscious this choice is biased, Supergravity Theory is a theory:

- that, in the context of Supersymmetry, forces to consider spinor fields as central objects;
- for which an important ingredient is a differential 3-form, an object that has not been historically so systematically explored by mathematicians;
- in which dimension 11 (always to be remembered as $4 + 7$) plays a specific role;
- in which Killing spinors play a central role;
- hence that connects naturally to metrics with special holonomy.

2. The View of a Mathematician on Spinors

Spinors as Algebraic Objects

As you all know, Élie CARTAN is the one who spotted in 1913 that usual algebraic objects (vectors, tensors) were not enough in the context of classification of representations of the orthogonal group. The approach I follow is based on the use of Clifford algebras on vector spaces endowed with a scalar product:

- Even-dimensional Clifford Algebras are endomorphism algebras of the vector space of *spinors*: $Cl(V,g) = \mathrm{End}(\Sigma V)$; and odd-dimensional Clifford algebras direct sums of two endomorphism algebras of vector spaces of *spinors*;
- The Spin group Spin_n, a 2-fold cover of SO_n, can be viewed as a subgroup of the multiplicative group of Clifford algebras, and therefore acts (irreducibly) on the space of spinors;
- Spinors for different metrics can be compared through an explicit construction found as late as 1992;
- When vectors and differential k-forms have a specific "dimension", to specify the "dimension" of a spinor is trickier.

Spinors as Fields

One needs to consider spinors in the context of manifolds, something that has not been so evident, as the last statement in the book by Élie CARTAN *"La théorie des spineurs"* warned that the usual construction using local coordinates was problematic. This difficulty can be overcome using Bundle Theory. The following is now well known:

- An oriented manifold M can be endowed with a Spin_n-principal bundle covering $SO_g M$ for a Riemannian metric g if and only if $w_2(M) = 0$;
- Via the associated bundle construction, once a Spin_n-principal bundle γ is chosen, one forms the spinor bundle $\Sigma^\gamma M \longrightarrow M$;
- The spinor bundle $\Sigma^\gamma M \longrightarrow M$is naturally endowed with a covariant derivative D^γ;
- Spinor fields are of course sections of the spinor bundle.

Natural Operators on Spinor Fields

On the space of spinor fields only two first order differential operators are universally defined, and combinations thereof:

- This is because $T^*M \otimes \Sigma_\gamma M$ decomposes into two invariant subspaces, a copy of $\Sigma_\gamma M$ and another space $\Sigma_\gamma^{3/2} M$;
- The projection onto the first factor gives rise to the *Dirac operator* $\mathcal{D}$ that maps spinor fields to spinor fields, and is defined, for a spinor field ψ, by
$$\mathcal{D}^\gamma \psi = \sum_{i=1}^{n} e_i . D^\gamma_{e_i} \psi \ ,$$
where (e_i) denotes an orthonormal basis of the tangent space;
- The projection onto the second factor gives rise to the *twistor operator* P, which maps $\Gamma(\Sigma_\gamma M)$ to $\Gamma(T^*M \otimes \Sigma_\gamma M)$, and is defined for $X \in TM$ and ψ is a spinor field by
$$(P^\gamma \psi)(X) = D^\gamma_X \psi + \frac{1}{n} X . \mathcal{D}^\gamma \psi \ .$$

Dirac Operators

Here are some important properties of the Dirac operator, well known to you of course:

- It is *a square root of the Laplace-Beltrami operator*, hence an elliptic operator in a Riemannian setting;
- Its principal symbol is given by Clifford multiplication;
- It is self-adjoint;
- In even dimensions, it exchanges the chirality of spinors, hence non-trivial eigenspinors for the Dirac operator have necessarily components of both chiralities, unless they are in its kernel.
- For a spinor field ψ, the *Schrödinger-Lichnerowicz formula* reads
$$(\mathcal{D}^{\gamma})^2\psi = (D^{\gamma})^* D^{\gamma}\psi + \frac{1}{4}\, Scal_g\, \psi \ ,$$
where $(D^{\gamma})^*$ denotes the adjoint of the covariant derivative D^{γ} and $Scal_g$ the *scalar curvature* of g.

3. Killing Spinors

Killing Spinors

The notion of a *Killing spinor* defined below was introduced by Roger PENROSE and Martin WALKER while they were looking for first integrals of the geodesic flow of the Kerr metric.

Definition (R. PENROSE)

A Killing spinor ψ *is a spinor field lying in the kernel of* $\mathcal{P}$ *and an eigenspinor for* $\mathcal{D}$*. Its characteristic equation is, for some* $\lambda \in \mathbb{C}$*,*

$$\forall X \in TM, \; D_X\psi + \frac{1}{n}\lambda\, X.\psi = 0 \; .$$

- Killing spinors are some sort of infinitesimal supersymmetry.
- The 1-form ξ_ψ defined on $X \in TM$ by $\xi_\psi(X) = (X.\psi, \psi)$ is dual to a *Killing vector field*, i.e. an infinitesimal isometry.
- Other components of $\psi \otimes \bar{\psi}$ also satisfy interesting conditions.
- The curvature tensor of D^γ acting on ψ is very special, namely, for all X, $Y \in TM$, $R_{X,Y}\psi = \lambda^2/n^2\,(X.Y - Y.X).\psi$.
- It follows that $Ric_g = 4\,\lambda^2\,(n-1)/n^2\,g$.

Killing Spinors (continued)

It is useful to discuss cases according to the eigenvalue λ:

- If $\lambda = 0$, then the Killing spinor is parallel, and hence the metric has *reduced holonomy* (see later);
- If $\lambda \in i\,\mathbb{R}^*$, then M is non compact;
- If $\lambda \in \mathbb{R}^*$, then the Ricci curvature is uniformly positive, and by Myers' Theorem, M is compact if the metric g is complete.

The key construction, due to Christian BÄR, goes as follows:

- Construct the cone $CM = M \times \mathbb{R}^{+*}$ over M and endow it with the cone metric $\bar{g} = dr^2 + r^2\, g$;
- Then, through an identification of an action of the group Spin_{n+1} within the Clifford algebra $Cl_g(M)$, map spinor fields on M into spinor fields on CM;
- Through this identification, Killing spinors on M are mapped to parallel fields on CM.

A View via Modified Connections on Spinor Fields

Actually it is not surprising that Killing spinors can be related to parallel fields (on some auxiliary space).

The link comes from modified connections on spinor fields:

- Indeed the equation that is satisfied by Killing spinors can be viewed as that of a spinor field parallel for the modified connection $D^{\lambda}_X = D_X + \frac{1}{n}\lambda X$;
- Given any 1-form A taking its values in the endomorphisms of the spinors, such a construction can be generalised to modified connection D^A defined on a spinor field ψ and $X \in TM$ as $D^A_X(\psi) = D_X\psi + A(X)(\psi)$;
- Of special interest in relation with Supergravity is to take A to be the fundamental 3-form viewed as a 1-form with values in exterior 2-forms as 2-forms act naturally on spinors.
- If a non-vanishing A-modified Killing spinor exists, the metric g must satisfy the modified Einstein equation with right hand side given by the coupling to A.

4. Holonomy groups

The Concept of Holonomy

The concept of a *holonomy group* was introduced in Geometry by Élie CARTAN in 1925 and developed in the article *"Les groupes d'holonomie des espaces généralisés"* in 1926. The notion makes sense for any bundle endowed with a connection over a space.

Here are some key facts:

- Holonomy transformations are generated by parallel transport along closed curves with respect to the given connection;
- Armand BOREL and André LICHNEROWICZ established in 1952 that the group generated by such transformations is a Lie subgroup of the structure group of the bundle;
- Warren AMBROSE and Isadore M. SINGER showed in 1953 that the holonomy group relates to the curvature of the connection as its Lie algebra is generated by curvature transformations;
- Actually the holonomy group has a topological meaning as it is the smallest group to which the bundle can be reduced.

Holonomy groups in Riemannian Geometry

In the context of Riemannian Geometry, further results concerning the holonomy have been obtained for the Levi-Civita connection. Here are the key results:

- As the metric g is parallel under the Levi-Civita connection, the holonomy group is a subgroup of O_n if the manifold M is of dimension n, and of SO_n if M is oriented;
- Marcel BERGER gave the list of possible holonomy groups in the case of irreducible non symmetric spaces.
- Here is the list (in the oriented case): SO_n (the generic case), U_m, SU_m, Sp_q, $\mathrm{Sp}_1 \times \mathrm{Sp}_q$, G_2, Spin_7 and Spin_9 (shown by Alfred GRAY as not possible);
- It is interesting to note that André WEYL in a lecture at the Séminaire Bourbaki in 1962 does signal the importance of the results of Marcel BERGER.

Special Holonomy

Metrics with special holonomy in Berger's list have been the focus of a lot of attention by Riemannian geometers as constructing examples of some of them required great ingenuity.

Here is the situation now:

- Metrics with holonomy U_m are Kählerian, the Kähler form being the extra parallel object;
- Metrics with holonomy SU_m are Kählerian with a parallel complex volume form, hence the vanishing of the first Chern class and flat Ricci curvature; the first non-trivial examples came from the solution of the Calabi conjecture by YAU Shing Tung in 1976;
- The first compact examples of metrics with holonomy G_2 (in dimension 7) and Spin_7 (in dimension 8) were constructed by Dominic JOYCE and represents a real "tour de force" but many other examples were later found.

Special Holonomy G_2

Metrics with holonomy G_2 are particularly interesting as they mix in a very interesting way algebraic and differential properties which are also relevant to some questions in Theoretical Physics.

Here are some highlights:

- Metrics with holonomy G_2 are Ricci-flat and the manifold bearing them must have finite fundamental group;
- What plays an important role in discussing such situations is the 3-form whose stabiliser is G_2; it actually defines a metric at each point since G_2 is a subgroup of the orthogonal group, and this metric has special properties that follow from the fact that this fundamental form and its Hodge dual are closed;
- Similar considerations can be presented in dimension 8 with a 4-form to give rise to $Spin_7$ holonomy groups;
- There are many contributions to the discussion of special holonomy groups by of Robert BRYANT.

A Flashback on Manifolds with Killing Spinors

Many examples of manifolds admitting Killing spinors are connected to holonomy considerations:

- Nigel HITCHIN has shown that manifolds with parallel spinors have necessarily reduced holonomy SU_m, $Spin_7$ and G_2;
- Manifolds with imaginary Killing spinors (actually the case forced if the manifold is complete and non compact) have been classified by Helga BAUM: hyperbolic spaces or special warped products of $\mathbb{R}$ with a manifold with a parallel spinor;
- The classification for real Killing spinors is a bit more involved;
- For $2 \leq n$ the standard sphere has only one spin structure. For the standard metric, the bundle of spinors is trivialized by Killing spinors that are induced by parallel spinors in $\mathbb{R}^{n+1}$ as suggested by the construction of Christian BÄR;
- In even dimensions other than 6, only standard spheres carry Killing spinors. In dimension 6, one also finds the manifolds endowed with a nearly Kähler non Kähler metric.

A Flashback Manifolds with Killing Spinors (cont.)

Other examples are interesting from a geometric or even a physical point of view:

- In dimension 5, Thomas FRIEDRICH and Ines KATH proved that one finds only manifolds with an Einstein-Sasaki metric; this was generalized by BÄR in all dimensions $4q+1$;
- In dimensions $4q+3$ (for $2 \leq q$), BÄR showed that one has to add the Sasaki 3-manifolds, a case studied further by Andrei MOROIANU;
- In the remaining dimension 7, the extra family to add corresponds to manifolds for which the cone built over them carries a metric with Spin_7 holonomy;
- The so-called *squashed 7-sphere* is a very interesting metric on S^7: it is a non-standard Einstein metric with a very geometric description but also a connection to supergravity.

5. Perspectives from a Spinorial Point of View

Developing a Spinorial Geometry

As you have by now found out, the consideration of spinor fields and natural operators defined on them proved to be a powerful tool to deal with some specific important questions in Riemannian Geometry.

More is expected:

- A link to the Ricci curvature, and in particular to the Einstein condition, as it appeared in presence of a Killing spinor;
- A special role in metrics with special holonomy;
- So far, mathematicians focused attention on $\frac{1}{2}$-spinor fields;
- Attention should certainly also be devoted to spinors with higher spins, $\frac{3}{2}$ to start with, e.g. through a more systematic study of the *Rarita-Schwinger operator*. Mathematicians have not done that so far;
- Actually this gives me the opportunity to point to another article by Bernard JULIA relating the Rarita-Schwinger operator and moduli of Einstein metrics.

Another Article by Bernard JULIA

C. R. Acad. Sc. Paris, t. 295 (20 septembre 1982) Série II —113

PHYSIQUE THÉORIQUE. — Système linéaire associé aux équations d'Einstein. Note (*) de Bernard Julia, présentée par Claude Bouchiat.

Nous exhibons un système différentiel extérieur linéaire qui est « intégrable » au sens de E. Cartan si et seulement si les équations d'Einstein (dans le vide) sont satisfaites. Nous développons l'analogie avec le programme des twisteurs et avec les systèmes hamiltoniens qui admettent une paire de Lax que l'on appelle « complètement intégrables ». La supersymétrie apparaît à nouveau dans l'étude de systèmes réalistes.

Les progrès récents de la théorie des équations aux dérivées partielles résultent en grande partie de la découverte de systèmes linéaires « associés » qui admettent pour condition de « compatibilité » les systèmes non linéaires que l'on souhaite étudier. Il nous faut donc préciser la notion de compatibilité. Nous mentionnerons deux exemples bien connus : l'équation des twisteurs et la paire de Lax de l'équation de Sine-Gordon qui correspondent respectivement à la self-dualité du tenseur de courbure de Weyl W = 0 [1] et à l'équation $\Box\, a = \sin a$ [2].

Nous présentons un troisième système linéaire qui est intégrable au sens d'Elie Cartan [3] si et seulement si les équations d'Einstein pour la gravitation sans sources extérieures sont satisfaites, il s'agit tout simplement de la forme linéarisée des équations de propagation du champ de spin 3/2 de la supergravité dans un espace (pseudo) riemannien donné [4].

The Interface between Mathematics and Physics

As you know, the 2019 Breakthrough Prize led to a controversy:

- Some of it was related to the fact that, if Supergravity is to be honoured, all the main initiators should be honoured with the prize. For sure there are others than the ones distinguished, e.g. Stanley DESER and Bruno ZUMINO, but of course several others...
- Critical views were also expressed on chosing Supergravity as topic, *"a failed idea"* to some, leading to (abusive) statements such as *"The message is clearly that in fundamental physics contact to observation is no longer necessary."*
- Did not it take more than half a century to have some solid evidence that black holes exist?
- This gives me the opportunity to call your attention to a quote from the Committee of the Königlich Preussische Akademie der Wissenschaften that offered a position in Berlin to Albert EINSTEIN in 1913.

Preussische Akademie der Wissenschaften Quote

Berlin, 12. Juni 1913.

Die unterzeichneten Mitglieder der Akademie beehren sich, die Erwählung des ordentlichen Professors der theoretischen Physik an der eidgenössischen technischen Hochschule in Zürich, Dr. Albert Einstein, zum ordentlichen Mitglied der Akademie, mit einem besonderen persönlichen Gehalt <zunächst> von <6000> 12000 M., zu beantragen.[2]

in bemerkenswerter Weise Stellung genommen hätte. Daß er in seinen Spekulationen gelegentlich auch einmal über das Ziel hinausgeschossen haben mag, wie z. B. in seiner Hypothese der Lichtquanten, wird man ihm nicht allzuschwer anrechnen dürfen; denn ohne einmal ein Risiko zu wagen, läßt sich auch in der exaktesten Naturwissenschaft keine wirkliche Neuerung einfüh-

The statement was signed by Max PLANCK, Walther NERNST, Heinrich RUBENS and Otto WARBURG.

I thank you for your attention.

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